REVIEWS OF MODERN PHYSICS VOLUME 43, NUMBER 2, PART 1 APRIL 1971

Review of Recent Work on Narrow Resonance Models*

DENNIS SIVERS†
Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California
JOEL YELLIN
Institute for Advanced Study, Princeton, New Jersey

We review recent work on narrow resonance models. We take the point of view that such models play a role similar to that of the Lee model in quantum field theory, and that they cannot therefore be directly compared with experiment. Examples of various aspects of these models, including general self-consistency and the construction of amplitudes with external currents are reviewed, and the related diseases are listed. A critical discussion of narrow resonance phenomenology is given. Associated questions which seem to us suitable for further study are discussed and summarized.

CONTENTS

I. INTRODUCTION

We present here a review of recent work on crossing-symmetric narrow resonance models. In the past year, such models have been the subject of a burgeoning amount of research and we hope that our exposition will be an aid to those who want to acquaint themselves with these developments. In our discussion of the subject we have made no attempt to disguise our personal prejudices, but have instead tried to articulate them as clearly as possible.

It is our point of view that we are dealing here with a model, such as the Lee model (Lee, 1954) in field theory, rather than with a theory amenable to direct experimental test. For this reason, while we accept attempts to use clues gleaned from narrow resonance models to generate phenomenological forms, we are sharply critical of contentions that these constructions also embody tests of fundamental principles.

In our title we have referred to the "narrow resonance model." Although the Veneziano model is the best-known example of such a model, we want to distinguish it from the general class of narrow resonance models. By the Veneziano model (Veneziano, 1968) we mean a representation of a scattering amplitude by a sum over a small number of terms of the form

$$\frac{\Gamma(m-\alpha(s))\Gamma(n-\alpha(t))}{\Gamma(m+n+p-\alpha(s)-\alpha(t))}.$$

We believe that this distinction is useful since much of

* This work was supported in part by the U.S. Atomic Energy Commission.
† NSF Predoctoral Fellow; present address: High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439.

the work we discuss does not make use of that particular functional form. The $\Gamma\Gamma/\Gamma$ form is a simple example of a crossing-symmetric, Regge-behaved, dual, narrow resonance amplitude and, as such, can be used as a touchstone to test broader theoretical speculation. It is an interesting problem to determine under what circumstances the Veneziano model and the general narrow resonance model become equivalent.

We discuss the narrow resonance model within the context of what we call the "nondynamical" assumptions of Lorentz invariance, crossing symmetry, proper statistics, and consistency with the discrete symmetries. Furthermore, we assume consistency with internal symmetry, specifically isospin and/or $SU(3)$. To a large extent, our assumptions concerning internal symmetries cannot be tested within the context of the model. There exists as yet no reasonable argument which either singles out a particular internal symmetry or gives any insight into the mechanism by which symmetries are broken.

Predictions outside the scope of narrow resonance models are presumed to rely heavily on unitarity. At the level of our present understanding, the division of the properties of scattering amplitudes into categories labelled "dynamic" and "nondynamic" is only semantic. In this framework unitarity is generally assumed to be a dynamic property which can be treated separately, and many active research efforts involve attempts to "unitarize the Veneziano model" in the belief that unitarity can be invoked at some late stage to extrapolate from the narrow resonance limit in a well-defined way. Since the narrow resonance world is an artificially elegant one whose dynamical properties are made manifest by infinite strings of two-body resonances, it lacks crucial features known to be present in the physical world, and thus such a unitarization procedure is bound to be difficult, if not impossible.

The plan of this paper is as follows: In Sec. II we discuss the properties of the general crossing-symmetric narrow resonance model (CNRM) for a four-body amplitude. In Sec. III, we illustrate many of the points touched on in Sec. II using the simple Veneziano model for $\pi\pi$ scattering. Readers unacquainted with this general subject may prefer to read Sec. III first in order to orient themselves. In Sec. IV we discuss alternative narrow resonance models such as those suggested by Virasoro and Mandelstam. In Sec. V we discuss PCAC and current algebra in connection with the CNRM. In Sec. VI we touch on proposed schemes to avoid the narrow resonance approximation while retaining the other desirable properties of the Veneziano model. In Sec. VII we discuss the generalization of the Veneziano model to N-particle amplitudes, the problems of factorization, and the use of the narrow resonance amplitude as a Born term in a perturbation expansion. In Sec. VIII we discuss high-energy diffraction, the Pomeranchon, and duality in the general context of the CNRM. In Sec. IX we discuss attempts to form Reggeized Feynman diagrams containing closed loops. In Sec. X, we examine proposed phenomenological forms arising from the CNRM. In the final section, we summarize our remarks and list what we consider to be interesting unanswered questions deserving further attention.

The cutoff date of our general literature survey was 15 July 1969, and we have used the SLAC listing of preprints in particles and fields (PPF) to construct as nearly a complete set of references as possible. We have also tried to include more recent work which seemed to us relevant, and in the process we have certainly missed papers which may be of significance.

In our references we have used the following system. Review articles, physics texts, mathematics texts, and journal articles are compiled in four separate lists in the bibliography. A few older articles are referenced in footnotes. If there is a possibility of confusion, references to review articles are marked (*). References to books are marked with a dagger (†). Mathematics and physics texts are not distinguished in the text. Journal articles not otherwise referenced in the text are listed at the end of the section to which they are relevant. In the bibliography, journal articles are cross referenced with the section to which they are relevant.

II. GENERAL PROPERTIES OF NARROW RESONANCE AMPLITUDES

In this section we discuss certain general features of narrow resonance models for strong interaction scattering amplitudes. The reader unacquainted with the subject may find it more convenient to first read Sec. III, where a specific model for $\pi\pi$ scattering is discussed in detail.

In Table 2.1, we list a set of assumptions and properties which conveniently outlines the discussion to follow. In this section we will touch on the narrow resonance model's connection with internal symmetries, finite energy sum rules (FESR), Regge behavior, duality, and the interference model. The remaining items in Table 2.1 will be dealt with in Sec. III.

A. Internal Symmetry

The first four "kinematic" assumptions listed in Table 2.1: Lorentz invariance, consistency with crossing symmetry, the discrete symmetries, and Bose and Fermi statistics will be taken as given.[1] The assumption of consistency with internal symmetry necessitates some brief remarks.

The procedure used in forming a narrow resonance amplitude is first to choose a particular internal symmetry and associated representation or representations, and then construct the most general set of kinematic singularity free invariant amplitudes (Williams, 1963)

[1] Other viewpoints are possible. For example, see the remarks of Chew in Jacob and Chew† (1964), and in Chew† (1964). Calling crossing symmetry kinematic is our semantic choice.

or (alternatively) helicity amplitudes (Wang, 1966; Fox, 1967; Jackson and Hite, 1967; Arbab and Jackson, 1968; Cohen-Tannoudji, Morel, and Navelet, 1968; Mandula, 1968) consistent with these and with the other kinematic assumptions.

In general, there will be sets of solutions for each choice of internal symmetry group and related representations, as discussed in Sec. III for $\pi\pi\rightarrow\pi\pi$. The model itself gives no clue regarding which symmetry group or representation is to be preferred; neither does it determine the number of particles and/or conserved quantum numbers, nor the size and nature of the breaking of the assumed internal symmetry.[2]

Said in another way, the narrow resonance scheme is in some sense equivalent to an infinite set of linear relations between pole residues. Through factorization, the pole residues are bilinear in the coupling constants or vertex functions. We have, therefore, an infinite set of sum rules, and with the usual choice of linear Regge trajectories, these determine the relative sizes of all the coupling constants.

In order to make clear the limitations under which we are working here, it is important to note that there are three further important questions which cannot be answered in the context of such systems:

(A) What is the absolute normalization of amplitudes?

(B) How many prominent resonances are there?

(C) Given the nature of the narrow resonance approximation (NRA) point, can one truncate the set of sum rules and still derive approximately valid results?

In other words, in the context of narrow resonance models, it is not possible to predict the strength of the strong interactions, the energy at which amplitudes become smooth, or to identify the set of resonances which determines the properties of amplitudes at low energies. The infinite set of narrow resonance sum rules contains as a subset the relations considered by Gilman and Harari (1968) and by Weinberg (1968) (see also Cronström and Noga, 1970) to which the same limitations apply. Under certain hypotheses about the answers to (B) and (C) above, Weinberg (1968) has pointed out that it is possible to derive Lie algebraic statements about the vertex functions involved. The reader is referred to his paper, and that of Gilman and Harari (1968), for further details.[3]

The question of the nature of the relations necessary to answer (A)-(C), and to decide how a particular internal symmetry and its breaking occur, deserves further study. It has, for example, been hypothesized by Chew (1970) that the nonlinear constraints of multi-Regge unitarity will fix the number of mesons in an internal symmetry multiplet. [See Chew (1969) for related remarks in this connection.]

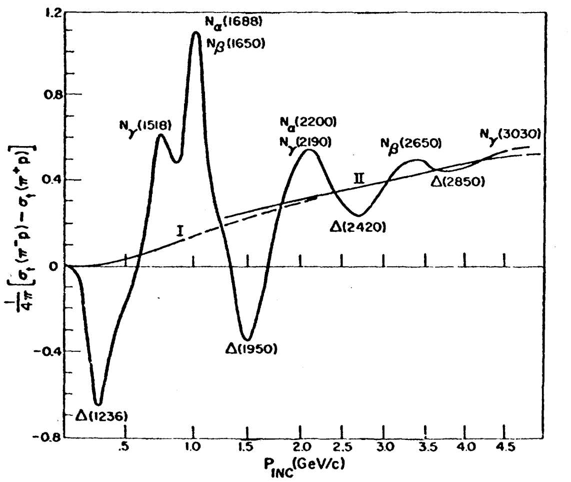

FIG. 2.1. Plot of k^2 (k is the center of mass momentum) times the difference of π^-p and π^+p total cross sections, showing evidence for approximate duality in finite energy sum rules. Curve I is k^2 times a nonresonant background extracted from πN phase shift analyses. Curve II is the extrapolation of the contribution of the ρ Regge trajectory. Curves I and II essentially are the integrand of the right-hand side of (2.2), while the oscillating data gives the integrand on the left. This approximate equality, in which the integrals, but not the integrands, match with the resonance contributions oscillating about the Regge term, is sometimes referred to as "semilocal duality." Figure taken from Chiu and Stirling (1968).

B. The Narrow Resonance Approximation and Finite Energy Sum Rules

In the narrow resonance approximation (NRA), we consider scattering amplitudes in which the familiar normal threshold branch points are absent and the resonance poles thought to be present on the second sheet of the physical amplitude occur on the real axis.

The possible dynamical importance of the NRA first became evident with the construction of the finite energy sum rule (FESR) bootstrap (Igi and Matsuda, 1967; Logunov, Soloviev, and Tavkhelidze, 1967; Ademollo *et al.*, 1968; Dolen, Horn, and Schmid, 1968; Mandelstam, 1968a; Schmid and Yellin, 1969).

The FESR's provide a realization of the infinite set of sum rules discussed above, and we review their formulation here.

Provided an amplitude satisfies analyticity and crossing, and is Regge behaved, its discontinuity in the energy variable $\nu=\frac{1}{2}(s-u)$, at fixed t, $D_\nu(\nu, t)$, satisfies the exact relation

$$\frac{1}{2}\int_{-N}^{+N} D_\nu(\nu,t)\nu^n\,d\nu=\frac{1}{2}\int_{-N}^{+N}\{\text{background integral}+\sum_j(\text{Regge cuts})_j+\sum_j(\text{Regge poles})_j\}\,\nu^n\,d\nu. \quad (2.1)$$

This exact expression can be greatly simplified if we

[2] This should be hedged slightly. See the remarks of Schmid and Yellin (1969), Sec. III.

[3] We would like to thank F. Gilman and S. Weinberg for very helpful discussions regarding the points raised above.

TABLE 2.1 General properties of narrow resonance models.

Kinematic	Lorentz invariance Crossing Symmetry Bose statistics Discrete symmetries Internal symmetry
Dynamic	Narrow resonance approximation
Secondary input and/or properties	1. Regge behavior 2. Infinite number of poles 3. Atonous duality 4. No equivalent interference model 5. Even spacing of poles 6. Polynomial residues 7. Positivity of widths 8. Uniqueness 9. Wrong signature fixed poles in J plane 10. Exponential behavior in exotic directions 11. Nonexistence of partial wave dispersion relations 12. Equivalence to Veneziano model 13. Absence of exotic resonances 14. Exchange degeneracy

make the following rather strong assumptions[4]:

(i) $\sum_j$ (Regge cuts)$_j \cong 0$;
(ii) background integral$\cong 0$;
(iii) $\sum_j$ (Regge poles)$_j \cong$leading pole only;
(iv) $\text{Im}\,\alpha(t)\cong 0$ and $\text{Re}\,\alpha(t)\cong a+bt$;
(v) $D_\nu(\nu, t)$ can be approximated by narrow resonances.

With these assumptions, the relation (2.1) is truncated to read

$$\frac{1}{2}\int_{-N}^{+N}\{D_\nu(\nu,t)\}_{\text{resonances}}\nu^n\,d\nu \cong \frac{\beta(t)N^{a+bt+n+1}}{a+bt+n+1}. \quad (2.2)$$

Equation (2.2) provides a consistency relation between the parameters of the leading Regge trajectory and the prominent resonances, and is likely to be valid only in an average sense, as illustrated in Fig. 2.1 in which the integrands of (2.2) are shown. The construction of the FESR for $\pi+\pi\rightarrow\pi+\omega$ (Ademollo *et al.*, 1968) led Veneziano (1968) to the form he suggested for the narrow resonance model, and (2.2) yields the set of sum rules referred to above.

In the models we will discuss, (i), (iv), and (v) are exact statements, and by a clever choice of t and N the background integral (ii) can also be neglected. Statement (iii), on the other hand, does not hold in these models since there are nonleading contributions on both sides of the FESR. In particular, the expression giving the high-energy behavior of the amplitude in terms of an infinite number of Regge pole terms is, in general, only an asymptotic expansion and not a convergent sum. Only for $\alpha(t)=$integer does the Regge series in the model converge. In this case there are a finite number of terms on the right-hand side of (2.1).[5]

In the exact relation (2.1) there are necessarily pieces which account for high-energy elastic diffraction scattering. As we will see in Sec. III, such terms, usually lumped together and called the Pomeranchon, cannot readily be accommodated in a narrow resonance model [see Wong (1969a) for an opposing view]. This dovetails nicely with the hypothesis of Freund (1968a) and Harari (1968), who equate the contributions of the Pomeranchon trajectory to the right-hand side of (2.1) to nonresonant background on the left. Since narrow resonance amplitudes have no background, the truncated FESR (2.2) are popularly supposed to hold only for those amplitudes which do not couple strongly to the Pomeranchon. This assumption has not been well explained, and at present it has only a rather striking empirical significance (Gilman, Harari, and Zarmi, 1968; Harari, 1969).

Clearly, there is an as yet unknown relation between the mysterious nature of the Pomeranchon, and the answers to (A)–(C) above. In fact, it is a reasonable guess that the Pomeranchon is associated with the existence of the infinity of inelastic channels, and that it therefore is an essential aspect of unitarity, which is conspicuously omitted from the list of assumptions in Table 2.1. Narrow resonances on the real axes of the Mandelstam variables violate unitarity. We therefore are not investigating a complete theory, but a model with a serious flaw. One can, for example, compare the formulation of the narrow resonance model with that of the N/D model (Chew and Mandelstam, 1960), which preserves elastic unitarity but violates crossing.

In the authors' opinion, the unitarity violation of the narrow resonance model totally precludes any practical applications whatsoever. This view is not generally

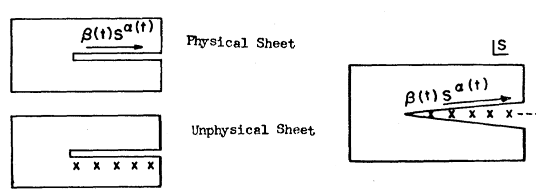

FIG. 2.2. Analytic structure of an amplitude with physical cuts vs a narrow resonance amplitude. The physical amplitude is power bounded on the physical sheet, while the narrow resonance amplitude has no sheet structure and has unbounded asymptotic behavior along the line of poles, unless one goes a finite angle into the complex plane.

[4] The arguments below are taken from Schmid and Yellin (1969) and define the FESR in a rather strict sense which precludes the phenomenological applications of continuous moment superconvergence relations considered for example by Liu and Okubo (1967), Olsson (1967), and Barger and Phillips (1968). One of us (J.Y.) would like to thank K. Raman for emphasizing this point in a private communication.

[5] In general, there will be an infinite number of terms on the right-hand side of (2.2). However, at $\alpha(t)=$integer, the gamma functions which appear in $\beta(t)$, in the model, ensure that only a finite number of Regge trajectories contribute.

subscribed to. For example, it has been hypothesized by Chew (1969) that limitations on narrow resonance models due to unitarity violation can be avoided by using exterior physical inputs derived from other models. We will return repeatedly to the question of the realistic interpretation of narrow resonance models below.

C. Regge Behavior

We now turn to the secondary inputs and/or properties listed in Table I.[6] We will insist that the narrow resonance model has Regge asymptotic behavior, by which we mean that the amplitudes behave like

$$\lim_{|s|\to\infty,\ t \text{ fixed}} A(s,t)=\beta(t)s^{\alpha(t)} \tag{2.3}$$

uniformly in the entire complex s-plane except in a direction along the line of poles on the real s axis. The elimination of this one direction in the complex s-plane from the restriction of Regge asymptotic behavior (Veneziano, 1968) seems to us a reasonable restriction in view of the absence in the model of normal threshold branch cuts. Fig. 2.2 illustrates the asymptotic behavior on the physical sheet of the s-plane for an amplitude with physical cuts, and for a narrow resonance amplitude. In this picture a wedge around the real axis in the narrow resonance model can be viewed as mimicking the properties of a second, nonphysical sheet in the more realistic amplitude. The physical region of the narrow resonance amplitude can be viewed as the area above the upper boundary of this wedge, just as the physical region of an amplitude with cuts is taken as the area above the boundary of the cut.

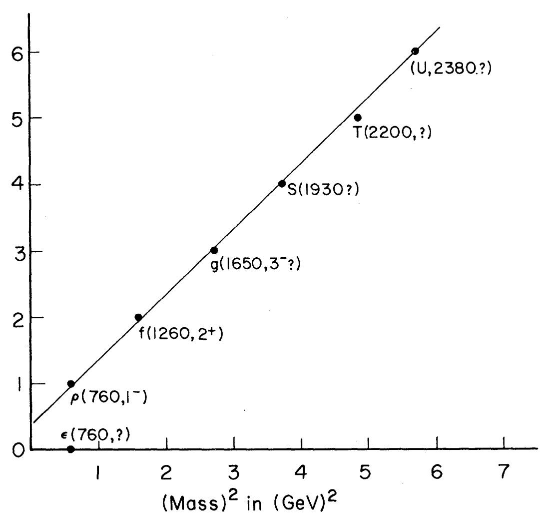

FIG. 2.3. Chew–Frautschi plot showing states lying on the degenerate ρ, f trajectory used in the Veneziano model. The parameters of the resonances in the S, T, and U regions are highly speculative.

[6] We have hedged the title of the list because several different, but almost equivalent, sets of assumptions are in general use. This will be discussed further in Secs. II.E and III.N.

As for property 2 in Table 2.1, the mathematically oriented reader may have already observed that the asymptotic behavior (2.3) in the absence of cuts already requires that we consider amplitudes with an infinite number of poles. If we write an amplitude with a finite number of poles in the form

$$A(s,t)=\sum_{k=0}^{N}\frac{c_k(t)}{s-s_k}+E(s,t), \tag{2.4}$$

where $E(s,t)$ is entire in s and $c_k(t)$ is a polynomial in t, then the finite sum in (2.4) will have fixed power behavior

$$\sum_{k=0}^{N}\frac{c_k(t)}{s-s_k}\underset{s\to\infty}{\longrightarrow}O(1/s) \tag{2.5}$$

in the asymptotic region. Picard's second theorem on essential singularities (Titchmarsh,† 1939, Sec. 8.8) guarantees that no entire function except a polynomial has uniform power behavior, so $E(s,t)$ cannot exhibit Regge behavior by itself. Also, there cannot be cancellations between $E(s,t)$ and a finite number of pole terms to produce the asymptotic form (2.5).

In order to have an infinite number of poles without an accumulation point in the finite plane, we must have

$$\lim_{k\to\infty}|s_k|=\infty, \tag{2.6}$$

and the assumed identification of the location of the poles in the amplitude with positive integral values of the s-channel Regge trajectory,

$$\alpha(s_k)=k, \qquad k=0,1,2,\cdots, \tag{2.7}$$

means we have infinitely rising Regge trajectories (Mandelstam, 1968a). Consistency with current experimental (cf. Fig. 2.3) suggests we should consider linear Regge trajectories[7]

$$\alpha(s)=a+bs. \tag{2.8}$$

D. Duality

We would like to discuss the subject of "duality" in terms of an amplitude with poles in two channels, s and t, which is symmetric under $t\leftrightarrow s$. The term "duality" was first invented by Chew and Pignotti (1968) to describe the observation of Dolen, Horn, and Schmid (1968) that there exist intermediate energies where some FESR's can be saturated on the left-hand side by a few dominant resonances, and on the right-hand side by the leading Regge trajectory, as discussed above. Controversy has arisen regarding the definition and applicability of "dual" and "interference" models (Barger and Durand, 1968). Most of the controversy is due to the ambiguities involved in dividing an amplitude

[7] It is not clear that nonlinear Regge trajectories should be excluded. See Capra (1969) and Coon (1969).

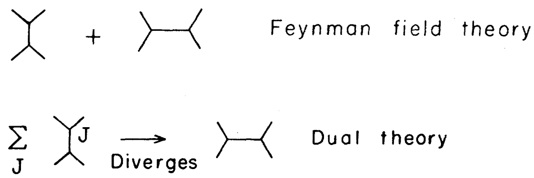

Fig. 2.4. Comparison between Lagrangian field theory and dual tree diagram model. In the field theory, diagrams containing poles in overlapping channels are added. In a dual theory the sum over poles in one channel diverges to produce poles in overlapping channels; if sums from both channels are added, double counting occurs.

with cuts into "resonances" and "background" and the related difficulties in measuring resonance parameters from Argand diagrams (Jackson,* 1970). For the details of the arguments, the reader can consult Allesandrini, Amati, and Squires (1968); Chiu and Stirling (1968); Durand (1968); Collins, Ross, and Squires (1969); Donnachie and Kirsopp (1969); Jengo (1969); Schmid (1969a).

Since NRA amplitudes, by definition, contain no background, the situation is much clearer and we can better understand the nature of duality. First, we distinguish between two possibilities. Suppose we write

$$A(s,t)=\sum_{k=0}^{\infty}\frac{c_k(t)}{s-\xi_k}+\sum_{k=0}^{\infty}\frac{c_k(s)}{t-\xi_k}+E(s,t), \quad (2.9)$$

where the sum over $s(t)$ poles converges for all $t(s)$, where the ξ_k are a positive set of constants, ordered monotonically in k, and where the c_k are polynomials. $E(s,t)$ is a symmetric function entire in both s and t. Equation (2.9) is the narrow resonance form of the interference model, where t-channel poles and s-channel poles are added separately as would be the case in a Feynman field theory[8] (see Fig. 2.4).

This contrasts with what we will call "atonous" duality[9] (Sivers and Yellin, 1969b), where

$$A(s,t)=\sum_{k=0}^{\infty}\frac{g_k(s)}{t-\xi_k} \qquad (\text{for } s<\xi_0) \quad (2.10\text{a})$$

and also

$$A(s,t)=\sum_{k=0}^{\infty}\frac{g_k(t)}{s-\xi_k} \qquad (\text{for } t<\xi_0) \quad (2.10\text{b})$$

and there is no arbitrary entire function. The sum over t-channel poles diverges for $t\geqslant\xi_0$ to give the s-channel poles and vice versa.

[8] In this connection see Abarbanel (1970), who attempts to write down a dual Lagrangian field theory incorporating the analytic properties of Eqs. (2.10) below. It seems to be impossible to formulate such a theory without introducing nonlocal interactions, unless one is willing to accept the manifold difficulties to be discussed in Secs. VII and IX.

[9] This particular terminology was suggested to one of us (J.Y.) by Professor Y. Ne'eman. Compare the expressions (3.18) and (3.19) with those of Durand (1968). The ingredient which is absent in older treatments is the atonous duality statement that everything is determined in the narrow resonance limit if one knows the location of poles and their residues. This implies that the divergence of the series of cross-channel poles generates direct-channel poles, and vice versa.

The crucial point here is that the interference model form of $A(s,t)$, (2.9), is not possible if we demand that the sum over t-channel poles, which is assumed to be entire in s, also have Regge asymptotic behavior in s, as in (2.3). This follows, as discussed before, from Picard's theorem. Even if we exclude a wedge, $|\arg s|\geqslant\delta$, from the requirement of Regge asymptotic behavior, the limitation of the sum over t-channel poles to be an entire function in s of finite order and type prohibits the interference model form (Oehme, 1969a).

Clearly the only possibility is that neither of the two sums in (2.9) have Regge behavior in s, but that they separately have some sort of complicated behavior which cancels to produce Regge behavior. This is precisely what happens in an atonous dual amplitude. In order to decouple the channels and create a genuine interference model, it is necessary to go beyond narrow resonances and introduce cuts (Jengo, 1969).

We will illustrate the cancellation mechanism involved by considering the Beta function, $B(-\alpha(s),-\alpha(u))\equiv B(-x,-y)$. We will split up $B(-x,-y)$ just as do Lichtenberg, Newton, and Predazzi (1969), whose interpretation of the results is diametrically opposed to ours. We have

$$\begin{aligned}B(-x,-y)&=\int_0^1 du\,u^{-x-1}(1-u)^{-y-1}\\&=\int_0^\lambda du\,u^{-x-1}(1-u)^{-y-1}\\&\qquad+\int_\lambda^1 du\,u^{-x-1}(1-u)^{-y-1} \quad (2.11)\\&=B_\lambda(-x,-y)+B_{1-\lambda}(-y,-x) \quad (2.12)\\&=(-\lambda^{-x}/x)\,{}_2F_1(-x,1+y;1-x;\lambda)\\&\quad+[-(1-\lambda)^{-y}/y]\,{}_2F_1(-y,1+x;1-y;1-\lambda), \quad (2.13)\end{aligned}$$

where $B_\lambda(p,q)$ is the incomplete Beta function and ${}_2F_1$ is Gauss's hypergeometric function. The integral in (2.11) has end-point singularities at 0 and 1 which account for the x and y poles; the x poles are associated with $u=0$, the y poles with $u=1$. Therefore $B_\lambda(-x,-y)$ contains x poles and is entire in y, while $B_{1-\lambda}(-y,-x)$ contains y poles and is entire in x.

Now, restricting ourselves so that the complex parameter $\lambda\neq 0$ or 1, we see that different choices of λ amount to changing $B_\lambda(-x,-y)$ and $B_{1-\lambda}(-y,-x)$ by entire functions.

Writing out the partial fraction expansion of B_λ, we have

$$B_\lambda(-x,-y)=y^{-1}\sum_{N=0}^{\infty}\frac{\Gamma(N+1+y)}{N!(N-x)\Gamma(y)}+E(x,y;\lambda), \quad (2.14)$$

where $E(x,y;\lambda)$ is entire in x and the sum converges for $y<0$. The sum diverges for positive y to produce poles which are cancelled by similar poles in $E(x, y; \lambda)$ since $B_\lambda(-x, -y)$ is entire in y. We therefore have a whole spectrum of functions $B_\lambda(-x, -y)$, for different values of λ, which have the same partial fraction expansion in terms of poles in s. Only one of these functions, for $\lambda=1$, is atonous dual by our definition and Eq. (2.10a), in that there is no extra entire function. The atonous dual function is the Beta function itself, $B_1(-x, -y)$, which contains cross channel poles which appear as divergences in its partial fraction expansions.

As $|y|\rightarrow\infty$ with x fixed (or vice versa), $B_\lambda(-x, -y)$ has Regge behavior in half the complex y plane. Which half it is depends on whether $|\lambda|$ is greater than or less than 1. The behavior of $B_\lambda(-x, -y)$ and $B_{1-\lambda}(-y, -x)$ for asymptotic values of their arguments is shown in Figs. 2.5 and 2.6. As can be seen there, for one of the directions $y\rightarrow\pm\infty$, x fixed, $B_\lambda(-x, -y)$ has Regge behavior, while $B_{1-\lambda}(-y, -x)$ has Regge behavior for one of the directions $x\rightarrow\pm\infty$, y fixed.

In other words, the x poles in $B_\lambda(-x, -y)$ lead to Regge behavior in half the y plane, while the y poles in $B_{1-\lambda}(-y, -x)$ lead to Regge behavior in half the x plane. In the non-Regge half-planes, the two functions blow up exponentially. In order to get Regge behavior for both directions $x\rightarrow\pm\infty$, y fixed, we need to sum the two functions and go back to $B(-x, -y)$. The two incomplete Beta functions interfere in such a way that the sum is Regge-behaved, except of course along the lines of poles, as discussed in Sec. II.C.

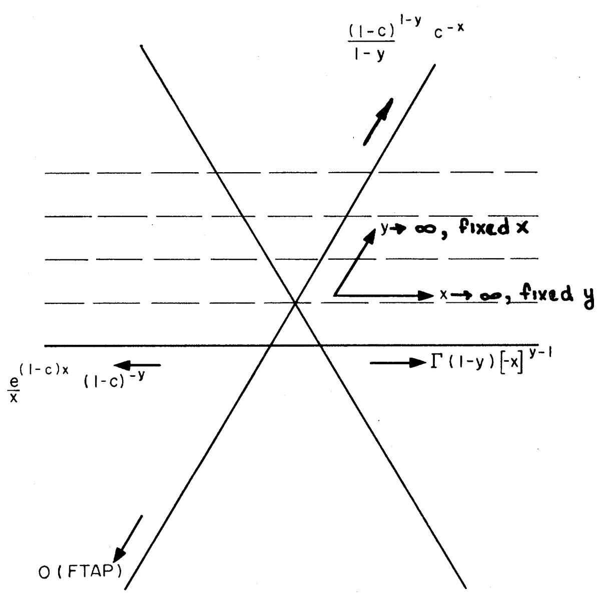

FIG. 2.5. Asymptotic behavior of $B_{1-c}(-y, -x)$, where Re$c>1$. The function has Regge behavior as Re$x\rightarrow+\infty$, but blows up exponentially as Re $x\rightarrow-\infty$. (The acronym FTAP means "faster than any power.")

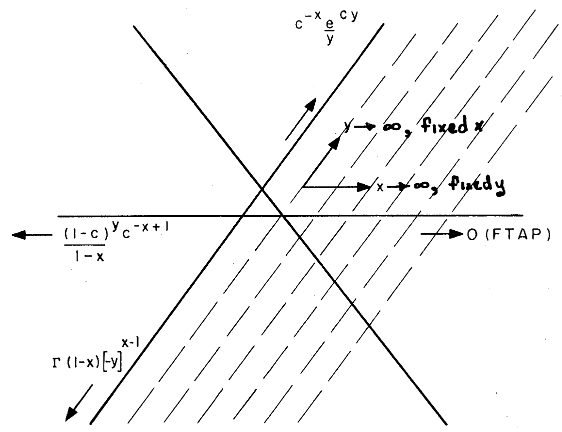

FIG. 2.6. Asymptotic behavior of $B_c(-x, -y)$ where Re$c>1$. The function has Regge behavior as Re$y\rightarrow-\infty$ but blows up exponentially as Re $y\rightarrow+\infty$. Note also the exponential behavior as Re$x\rightarrow-\infty$, which cancels out a similar exponential increase in $B_{1-c}(-y, -x)$ (cf. Fig. 2.5), so that the sum is Regge behaved in this region.

Lichtenberg *et al.* (1969) and Coulter, Ma, and Shaw (1969) identify the two pieces in (2.12) with the interference model breakup. We do not believe that a detailed cancellation of the type outlined above, between two terms, neither of which is acceptable as a physical amplitude due to the exponential blow up, is in the spirit of the original interference model (Barger and Cline, 1966; Barger and Cline, 1967; Barger and Durand, 1968), which depends on splitting the amplitude up into two terms in such a way that Regge behavior in $x(y)$ is associated with the $y(x)$ poles only.

For one of the directions $x\rightarrow\pm\infty$, y fixed, Regge behavior in x cannot be decoupled from the x poles in the narrow resonance model. In order to decouple Regge behavior from the direct channel poles, it is necessary to violate the narrow resonance approximation and introduce cuts. This is precisely what is done by Jengo (1969) in order to construct what he calls a generalized interference model.

The definition of atonous dual functions is not, of course, limited to crossing symmetric functions of two independent variables. Partial fraction expansions are the narrow resonance formulation of dispersion relations, and the absence of entire functions is equivalent to the absence of undetermined subtraction constants.

Any function with poles in two independent variables which is determined entirely by its partial fraction expansion in one variable is atonous dual.

As we will see in Secs. VII and IX, the concept of atonous duality can be readily generalized to N-variable functions having the singularities of Feynman trees, and to functions having the singularities of one Feynman loop. It is a characteristic common to all these prescriptions that the divergence of the expression in terms of one set of poles generates another set of poles. This is indicated schematically in Figs. 2.4 and 2.7.

Atonous duality, as stated, is a dynamical property

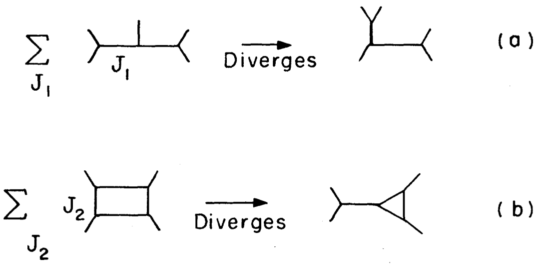

FIG. 2.7. Schematic sketch showing atonous duality as contained in the amplitudes discussed in Secs. VII and IX. The sum over poles in one invariant diverges to produce a pole in a crossed channel.

in that it places restrictions on the form which the residues of resonance poles can take. Not all functions of the form $\Gamma(p-x)\Gamma(q-y)/\Gamma(n-x-y)$ have atonous duality. When $n<p+q-1$, we can write

$$\frac{\Gamma(p-x)\Gamma(q-y)}{\Gamma(n-x-y)}=\frac{\Gamma(p+q-x-y)}{\Gamma(n-x-y)}B(p-x,q-y)$$
$$=(n-x-y)\cdots(p+q-1-x-y)$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(k-p+1+x)}{\Gamma(k+1)\Gamma(-p+1+x)}(q+k-y)^{-1},\quad(2.15)$$

which, in terms of the analogy between partial fraction sums and dispersion relations, can be viewed as a partial fraction sum with subtractions.

By beginning the poles in a gamma function form at very high energies—for example we may take $\Gamma(100-x)\Gamma(100-y)/\Gamma(100-x-y)$—it is easy to demonstrate that subtractions strongly affect the validity of an FESR like (2.2), for any reasonable choice of N.

E. The General Narrow Resonance Amplitude and Its Equivalence to a Sum over Veneziano Terms

We would like to comment on a very interesting discussion by Khuri (1969) regarding the construction of an absolutely convergent series of Veneziano terms. Specifically, Khuri considers a function of two variables, $F(x,y)$, and assumes the following:

(i) $F(x,y)$ is symmetric and meromorphic, with poles at $y,x=0,1,2\cdots$;

(ii) $F(x,y)$ has the (Regge) asymptotic expansion

$$F(x,y)\underset{y\to\infty,\text{fixed }x}{\sim}\pi\Gamma(-x)\sum_{K=0}^{\infty}(-y)^{x-K}a_K(x);\quad(2.16)$$

(iii) The coefficients $a_K(x)$ in (2.16) are entire in x. As $x\to N$, an integer, $a_K(N)=0$ for $K\geq N+1$ so that

$$a_K(x)=[\Gamma(K-x)/\Gamma(-x)]b_K(x);\quad(2.17)$$

(iv) As $x\to N$, an integer, the residue of the pole in $F(x,y)$ can be calculated from (2.16) and is

$$g_N(y)=\frac{(-1)^{N+1}}{\Gamma(1+N)}\sum_{K=0}^{N}a_K(N)(-y)^{N-K};\quad(2.18)$$

(v) There exist conditions on the growth of $b_K(x)$ in x and K sufficient to guarantee that the series

$$\bar{F}(x,y)\equiv\sum_{K=0}^{\infty}\sum_{J=0}^{K}C_K{}^J\frac{\Gamma(K-x)\Gamma(K-y)}{\Gamma(K+J-x-y)}\quad(2.19)$$

converges uniformly for some domain of x and is equal to $F(x,y)$ there.

The reader will notice the connection between Khuri's assumptions and properties 1, 2, 5, and 6 in Table 2.1. Positivity of partial widths, property number 7, is inserted by conditions on the Regge residues, $\beta_l(s)$, which are related to the b_K by the rather formidable relation

$$b_K(x)=\tfrac{1}{2}\sum_{j=0}^{K}\sum_{n=0}^{j}\frac{\beta_n(s)}{(4q^2)\alpha_n(s)}\frac{[2\alpha_n(s)+1](-1)^n}{\cos\pi\alpha_n(s)}$$
$$\times\frac{\Gamma[-\alpha_n(s)+j]}{\Gamma[-2\alpha_n(s)+j+n]}\frac{(4q^2)^{j-n}}{(j-n)!}b^{j-n}\frac{(-a)^{K-j}}{\Gamma(K-j+1)},\quad(2.20)$$

where $x=a+bs$, and $\alpha_n(s)=x-n$. The important point to notice about (2.20) is that it involves an alternating series so that the positivity condition is not easily implemented.

Khuri attacks the following problem: Given $a_K(x)$ as in (2.17), construct $F(x,y)$ as a sum of the form (2.19). He has been able to find bounds on the growth of $b_K(x)$ in (2.17) in order that the sum converge uniformly in x. In fact, for $b_0(x)$ the requirement is

$$|b_0(x)|<M2^{|x|},\quad(2.21)$$

where M is some fixed number.

In order to have positive partial widths, we also need a bound on the behavior of $b_K(x)$ for increasing K. This remains as an unsolved but interesting mathematical problem. Khuri has reduced this problem to a study of the solution to a certain finite difference equation. The reader is referred to his paper for details.

Matsuda (1969a) has attacked this problem from a slightly different angle. He makes the usual kinematic assumptions, assumes narrow resonances, puts in linear trajectories and the absence of exotics, and he also excludes cuts and right-signature fixed poles in the J-plane. Positivity is not included. He then shows that $F(x,y)$ can be expressed as a convergent series in Veneziano terms. However, it is not necessarily true that the resulting sum Reggeizes properly.

Matsuda (1969b) has also given an illustrative example in this connection. He takes

$$s(x,y;\lambda)=[\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y)]$$
$$\times{}_2F_1[-x,-y;\tfrac{1}{2}(1-x-y);\lambda],\quad(2.22)$$

which reduces to

$$\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y),\qquad\text{for }\lambda=0.$$

For $0<\lambda\leq1$, Matsuda's example satisfies Khuri's requirements. It also has an exponential decrease in

$(x-y)$ for fixed $(x+y)$ [if $x=\alpha(s)$ and $y=\alpha(u)$, this would be fixed t], related to the absence of exotic states, and its leading trajectory has positive widths. However, as we shall discuss further in Sec. III.N, there are good reasons for believing that (2.22) has an infinite number of negative widths on nonleading trajectories.

The positivity requirement is the crux of the problem. It can be argued, for example, Sivers and Yellin, 1969b, that positivity along with the other requirements of Matsuda (1969a) seems to force the Regge residues to blow up exponentially, violating the fixed J bound of Jones and Teplitz (1967).[10]

Additional material relevant to this section can be found in Bitar (1969b), Childers (1969), Jacob and Mandelbrojt (1969), Jacobs (1969), Joshi and Pagnamenta (1969), Oehme (1969b), Phillips and Ringland (1969), Swift and Tucker (1969), and Wong (1969b).

III. THE NARROW RESONANCE MODEL FOR $\pi\pi$ SCATTERING

In order to illustrate the main features of the narrow resonance world discussed in Sec. II, we would like to study in some detail a model of this type for $\pi\pi$ scattering. This particular reaction has been studied extensively from many points of view—for example, using N/D and current algebra. There are, on the other hand, indications (Mandelstam, 1968a; Schmid, 1968) that the $\pi\pi$ interaction may be roughly described by a narrow resonance scheme.

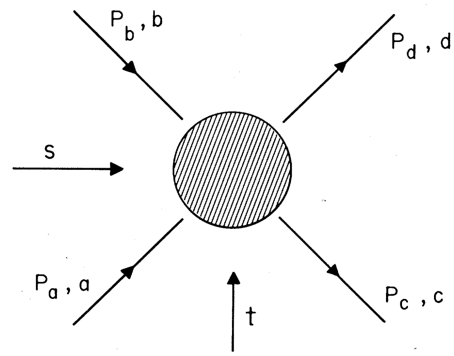

FIG. 3.1. The scattering process, $\pi_a\pi_b\rightarrow\pi_c\pi_d$.

A. Kinematic Requirements

The scattering process $\pi_a\pi_b\rightarrow\pi_c\pi_d$ is illustrated in Fig. 3.1. The constraints of Lorentz invariance, crossing symmetry, Bose statistics, and isospin invariance can be satisfied by writing the amplitude in the form (Chew and Mandelstam, 1960)

$$M_{dcba}(s,t,u)=A(s,t,u)\delta_{ab}\delta_{cd}+B(s,t,u)\delta_{ac}\delta_{bd}+C(s,t,u)\delta_{ad}\delta_{bc}, \quad (3.1)$$

where the subscripts on the Kronecker deltas stand for the charge states of the pion and the Mandelstam variables (s, t, u) have their conventional definition in terms of the four-momenta

$$s=(p_a+p_b)^2, \quad (3.2a)$$

$$t=(p_a-p_c)^2, \quad (3.2b)$$

$$u=(p_a-p_d)^2. \quad (3.2c)$$

Unless otherwise stated, in the rest of this section we will set $m_\pi^2=0$ so that $s+t+u=0$.

We will find it convenient to work with the t-channel isospin amplitudes

$$A_0^t=3B(s,t,u)+A(s,t,u)+C(s,t,u), \quad (3.3a)$$

$$A_1^t=A(s,t,u)-C(s,t,u), \quad (3.3b)$$

$$A_2^t=A(s,t,u)+C(s,t,u). \quad (3.3c)$$

Because of the constraints of crossing, the invariant amplitudes in (3.1) have the symmetry

$$A(s,t,u)=A(s,u,t)=B(t,s,u)=C(u,t,s), \quad (3.4)$$

so that specification of any one of A, B, or C determines the amplitude completely. Comparing (3.3) and (3.4), we see that we can also determine the amplitude completely by specifying either A_0^t or A_2^t. In this section we will work with the amplitude A_2^t as our basic function.

[10] We emphasize that the arguments of Sivers and Yellin (1969b) lack the rigor of those of Khuri and Matsuda. The Regge residues for a specific $\pi\pi$ model will be computed in Sec. III.J. There it will be shown that the leading Regge residue has a form which yields the asymptotic behavior

$$\beta_0(t)\underset{t\to\infty}{\sim}(1/4\pi)e^{3/2}(4/e)^{-\alpha(t)}\sim\exp[-\alpha(t)\ln(4/e)].$$

As t goes to ∞ along a wedge near the negative real axis, the residue has an exponential increase. Since $\beta_0(t)$ is an analytic function of t, and has an infinite string of zeros at $\alpha=-(3/2), -(5/2)\cdots$, Carlson's theorem tells us this exponential increase must occur, as pointed out by Jones and Teplitz (1967). Jones and Teplitz further remark that in a theory with infinitely rising trajectories at least one of the following set of assumptions, considered in a related context by Khuri (1967), must fail:

(i) the amplitude $A(s,t)$ is analytic in the cut s plane and is bounded for fixed t by

$$f(s)=c\exp(|s|^{\frac{1}{2}-\epsilon});$$

(ii) $A(s,z)$ is bounded by $f(s)$ for fixed z;
(iii) the Sommerfeld–Watson transformation of the partial-wave amplitudes $a(J,s)$ exists, and $a(J,s)$ is bounded by $f(s)$ for fixed J;
(iv) $\alpha(s)$ and $\beta(s)$ are analytic with a single cut from $s=4\mu^2$ to ∞, $\alpha(s)$ is polynomial bounded, and $\beta(s)$ is bounded by $f(s)$.

In the model, (i) and (iv) are satisfied by construction, but the fixed z and fixed J bounds in (ii) and (iii), and the $\beta(s)$ bound in (iv) fail. (The amplitudes blow up exponentially for fixed z in the unphysical region.) The bad asymptotic behavior of the partial wave amplitudes $a(J,s)$, in s, expresses the fact that the background integral, in the model, grows exponentially for large s and dominates the Regge series if one pushes the usual Sommerfeld–Watson contour to the left of $J=-1/2$. The presence or absence of satellite poles has led to considerable confusion in the literature. For example, Chu *et al.* (1968) attempted to generate a crossing symmetric model with only one leading trajectory. As was shown by Dolen, Horn, and Schmid (1968), and somewhat more rigorously by Mandula and Slansky (1968), this makes no sense, at least in a dual model. Mandula and Slansky went on to attempt to prove that even with an infinite family of parallel trajectories, a dual crossing symmetric model could not exist. As shown by Goebel (1968) and by explicit construction by Veneziano (1968), such a model does, in fact, exist.

B. Eigenfunctions of the Crossing Operator

Define

$$X^t \equiv \begin{pmatrix} A_0^t \\ A_1^t \\ A_2^t \end{pmatrix} \qquad (3.5)$$

and similarly for the s and u channels. From the remarks of Sec. III.A, there exists a crossing operator, Θ, where

$$X^s = \Theta X^t. \qquad (3.6)$$

The operator Θ is composed of a numerical matrix and an operator which switches s- and t-channel four-momenta. The numerical matrix is (Chew, 1961)

$$C_{st} = C_{st}^{-1} = \begin{array}{c c} \begin{array}{ccc} 0 & 1 & 2 \end{array} & \\ \begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} & \begin{array}{c} 0 \\ 1, \\ 2 \end{array} \end{array} \qquad (3.7)$$

where the rows and columns refer to the isospins in each channel. If we choose a function, $F(s, u)$, as a trial function for the amplitude A_2^t, then, by the remarks in Sec. III.A,

$$X^t = \begin{pmatrix} -\frac{1}{2}F(s, u) + \frac{3}{2}F(t, u) + \frac{3}{2}F(s, t) \\ F(t, s) - F(t, u) \\ F(s, u) \end{pmatrix} \qquad (3.8)$$

is an eigenfunction of the crossing operator Θ if $F(s, u) = F(u, s)$.

If we insist our isospin amplitudes are linear combinations of a particular symmetric function $F(x, y)$, in general Θ is such that there will be two linearly independent eigenfunctions with eigenvalue 1. One possible choice is (3.8). Another linearly independent choice is $F(s, t) = A(s, t, u)$, which yields, using (3.3) and (3.4), the eigenfunction

$$X^{t\prime} = \begin{pmatrix} 3F(s, u) + F(t, s) + F(t, u) \\ F(t, u) - F(t, s) \\ F(t, u) + F(t, s) \end{pmatrix}. \qquad (3.9)$$

Choosing $F(s, u)$ so that it has no poles in the physical t region, these two different eigenfunctions of the crossing operator have, as might be expected, different $SU(2)$ properties. The eigenvector (3.8) has an $SU(2)$ structure $\mathbf{3} \oplus \mathbf{1}$ while, by adding (3.8) and (3.9), we get a solution with no isospin 1 internal states:

$$X^{t\prime\prime} = [F(s, t) + F(s, u) + F(t, u)] \begin{pmatrix} \frac{5}{2} \\ 0 \\ 1 \end{pmatrix}, \qquad (3.10)$$

which corresponds to $\mathbf{1} \oplus \mathbf{5}$. These remarks can, of course, be easily extended to $SU(3)$. For example, (3.8) corresponds to $\mathbf{8} \oplus \mathbf{1}$, while (3.10) corresponds to $\mathbf{27} \oplus \mathbf{1}$ in $SU(3)$. The solution (3.10) is inconsistent with the classification of the experimentally observed low-mass resonances which communicate with the $\pi\pi$ channel (Rosenfeld, *et al.*,* 1969). However, if we have a solution with internal resonances appropriate to a nonet scheme, we can always add a function of the form (3.10) in order to incorporate high-mass, nonleading exotic resonances.

C. The Choice of $A_2^t(s, u)$

We will choose a basic function $F(x, y) = A_2^t(s, u)$ ($x = a + bs$ and $y = a + bu$) which has the following properties:

(a) It is symmetric, $F(x, y) = F(y, x)$, and meromorphic with simple poles at $x = 1, 2, \cdots$ and $y = 1, 2, \cdots$.

(b) It has Regge asymptotic behavior:

$$\lim_{x \to -\infty, \text{fixed } y} F(x, y) \sim \Gamma(1-y) \times [(-x)^y + C_1(y)(-x)^{y-1} + \cdots]. \qquad (3.11)$$

(c) The residue of the pole at $x = K$, $G(K, y)$, is a polynomial in y of order K.

(d) The residue $G(K, y)$ has positive Legendre coefficients

$$\mu_L^K = \frac{1}{2} \int_{-1}^{+1} G(K, a+bu) P_L(z) \, dz \geq 0, \qquad (3.12)$$

where $z = 1 + 2u/(K-a)$.

(e) $F(x, y)$ has no poles in the physical t channel. Our choice for $F(x, y)$ is

$$F(x, y) = g[\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y)] \equiv gF_0(x, y). \qquad (3.13)$$

We conjecture (3.13) is unique under the imposition of conditions (a)–(e) plus an additional assumption:

(f) For $m_\pi = 0$ and $a = \frac{1}{2}$ the amplitude is zero along $1 - x - y = 1 - 2a + bt = bt = 0$.

The general functional form $\Gamma\Gamma/\Gamma$ was first suggested by Veneziano (1968), and the application to $\pi\pi \to \pi\pi$ is due to Shapiro and Yellin (1970) and Lovelace (1968). Lovelace first suggested the connection of $a = \frac{1}{2}$ with the PCAC-current algebra zero, condition (f). We will return to the question of uniqueness in Sec. III.N.

D. Asymptotic Behavior

The asymptotic behavior of $F_0(x, y)$ is shown in Fig. 3.2. To compute the behavior in directions which cross poles, we need an averaging procedure. Consider, for example,

$$\lim_{|x| \to \infty, \text{fixed } y} \Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y) \sim \Gamma(1-y)(-x)^y + O[(-x)^{y-1}]. \qquad (3.14)$$

This result arises from the well-known asymptotic expansion (Tricomi and Erdelyi, 1951)

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \underset{|z|\to\infty}{\sim} \sum_{n=0}^{\infty} C_n(\alpha,\beta) z^{\alpha-\beta-n}, \tag{3.15}$$

where

$$C_0(\alpha,\beta)=1,$$

$$C_1(\alpha,\beta)=\tfrac{1}{2}(\alpha-\beta)(\alpha+\beta-1),$$

$$\vdots$$

$$C_n(\alpha,\beta)=n^{-1}\sum_{m=0}^{n-1}\left[\binom{\alpha-\beta-m}{n-m+1} - (-1)^{n+m}(\alpha-\beta)\beta^{n-m}\right]C_m(\alpha,\beta). \tag{3.16}$$

However, (3.15) is not valid if we travel along the negative real axis. The argument of x must satisfy $|\arg x| \le \pi-\delta$ $(\delta>0)$ in order to avoid the violent oscillations due to the line of poles.[11]

Taking $t=1-x-y$ and $\nu=\frac{1}{2}(x-y)$, we see that for t fixed and large $|\nu|$

$$F_0(x,y)=\frac{\Gamma(\frac{1}{2}+\frac{1}{2}t+\nu)\Gamma(\frac{1}{2}+\frac{1}{2}t-\nu)}{\Gamma(t)} \underset{|\nu|\to\infty}{\sim} \frac{\pi\nu^2}{\Gamma(t)\cos\pi\nu}, \tag{3.17}$$

which goes to zero faster than any power so long as we avoid the poles in $\cos\pi\nu$ by taking the asymptotic behavior along a line a finite angle away from the real axis. Using this result, we can see that the isospin amplitudes, (3.8), Reggeize along one of the axes of the Mandelstam plot only if the channel in which the fixed invariant is the energy contains resonances. The asymptotic behavior in a direction corresponding to a fixed value of an invariant is exponentially decreasing if the corresponding channel has no resonances. This behavior is illustrated in Fig. 3.2 for $F_0(s,u)$. Said another way, if an exotic y channel has no Regge trajectories to provide power behavior, $x^{\alpha(y)}$, then the amplitude falls faster than any power asymptotically.

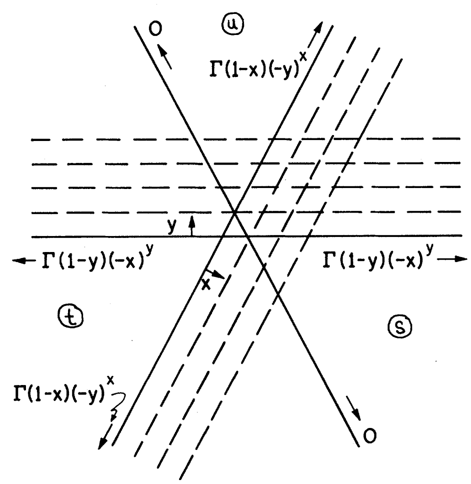

FIG. 3.2. Asymptotic behavior of the function $F_0(x,y)=\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y)$, where $x\propto s$, $y\propto u$, and 0 means exponential decrease.

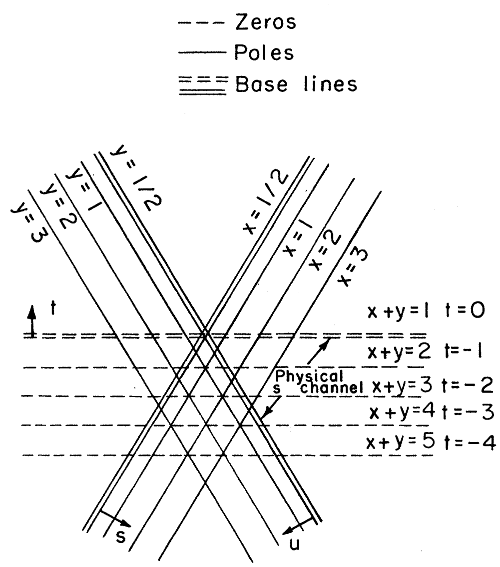

FIG. 3.3. Poles and zeros of

$$F_0(x,y)=\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y).$$

Poles are shown as solid lines, zeros as dotted lines. Except for the (PCAC) zero along $x+y=1$, all zeros serve to cancel possible double poles in the double spectral region $x>0$, $y>0$.

E. Duality: Formulation of Dispersion Relations

The function $F_0(x,y)$ is dual in the sense discussed above in Sec. II.D, its asymptotic behavior being entirely determined by its residues and the locations of its poles, with any additional entire function excluded.

In the language of (2.10), $F_0(x,y)$ can be represented as

$$\sum_{K=1}^{\infty}\frac{\Gamma(K+x)}{\Gamma(K)\Gamma(x)}(y-K)^{-1} \tag{3.18}$$

for negative x, and by

$$\sum_{K=1}^{\infty}\frac{\Gamma(K+y)}{\Gamma(K)\Gamma(y)}(x-K)^{-1} \tag{3.19}$$

for negative y. The sum of x poles diverges at positive y to form the sum of y poles. There is no additional entire function in (3.18) and (3.19). Expansions such as these are the narrow resonance formulation of dispersion relations, and the absence of entire functions is related to the absence of undetermined subtraction constants. One may also be interested in writing down the narrow resonance analog of the dispersion relation in ν for fixed t. Here this is (Whittaker and Watson,† 1927, Ch. 14.

[11] See in this connection the interesting example of Dolen, Horn, and Schmid (1968), Sec. VI.C.

TABLE 3.1. Coefficients of the Legendre polynomials contained in the model $\pi\pi$ amplitude (3.13), normalized to $L=N=1$.

	$N\rightarrow$	0	1	2	3	4
	4					1.43
	3				25/16	1.43
	2			3/2	25/16	0.681
	1		1	3/2	5/16	0.759
	0	0	1	0	5/16	0.0785
$\uparrow L$						

Ex. 24)

$$F_0(x,y)=\sum_{K=1}^{\infty}(-1)^K\frac{\Gamma(K+t)}{\Gamma(K)\Gamma(t)}\times\{[\nu+\tfrac{1}{2}(1-t)-K]^{-1}+[-\nu+\tfrac{1}{2}(1-t)-K]^{-1}\},\tag{3.20}$$

again with no additional entire function. From (3.20), it again is clear that the amplitude falls faster than any power as $|\nu|\rightarrow\infty$ for fixed t.

We can easily check that (3.19) diverges at $y=1$ in such a way that

$$F_0\underset{y\rightarrow 1}{\sim}x/(y-1).$$

Recalling the definition of the Riemann ζ function (EHF, 17.7) as

$$\zeta(z)=\sum_{K=1}^{\infty}K^{-z},\tag{3.21}$$

we have, evaluating (3.19) for high K in order to isolate the divergences,

$$\begin{aligned}\sum_{K=1}^{\infty}\frac{\Gamma(K+y)}{\Gamma(K)\Gamma(y)}(x-K)^{-1}&\underset{y\rightarrow 1}{\sim}[\Gamma(y)]^{-1}\sum_{K=1}^{\infty}\{K^y+\tfrac{1}{2}y(y-1)K^{y-1}+O(K^{y-2})\}\\&\qquad\times\{K^{-1}+xK^{-2}+O(K^{-3})\}\\&\underset{y\rightarrow 1}{\sim}-[1/\Gamma(y)]\{\zeta(1-y)+[\tfrac{1}{2}y(y-1)+x]\\&\qquad\times\zeta(2-y)+\cdots\}\underset{y\rightarrow 1}{\sim}[x/(y-1)],\end{aligned}\tag{3.22}$$

where we have used the fact that $\zeta(z)$ is analytic except for a simple pole of unit residue at $z=1$.

For a more detailed discussion of duality in narrow resonance models the reader is referred to Sec. II.D.

F. Poles and Zeros of $F_0(x, y)$

It is interesting to examine the poles and zeros of $F_0(x,y)$ over the Mandelstam diagram shown in Fig. 3.3. Note that there are no t-channel poles, that the poles are equally spaced, and that equally spaced zeros enter which cancel possible double poles in the double spectral region where s and u are both positive. There is also the extra (PCAC) zero along $t=1-x-y=0$, which we will discuss below.[12]

G. Angular Momentum Towers

Note in Fig. 3.3 the places where a dotted line crosses a solid pole line in the physical s channel ($y\leq\frac{1}{2}$, $t\leq 0$). The number of times a pole is crossed gives the order of its residue. For example, the pole at $x=2$ is crossed twice, at $t=0$ and $t=-1$. The corresponding residue has a factor $t(t+1)$, and since $\cos\theta_s$ is linear in t, we have a tower of poles at $x=2$ with angular momenta 0, 1, and 2. This structure is shown in Fig. 3.4. This degeneracy is the same as that of the Schrödinger hydrogen atom.

H. Behavior of Partial Widths

We define [cf. Eq. (3.19)]

$$H(N,L)=\int_{-1}^{+1}dzP_L(z)\frac{\Gamma(N+a+t)}{\Gamma(N)\Gamma(a+t)},\tag{3.23}$$

where $z=1+2t/(N-a)=\cos\theta$, the s-channel scattering angle.

The partial widths of the internal states in $F_0(x,y)$ are proportional to $H(N,L)$. In fact, one can convince himself that $H(N,L)\geq 0$ for all (N,L) provided $\frac{1}{2}\leq a\leq 1$. Frampton and Nambu (1969) have given an asymptotic argument, including an error estimate, that for $a\geq\frac{1}{3}$ the $H(N,L)$ are positive for large N. Numerically, up to rather high N, it is straightforward to show $H(N,L)>0$ (Shapiro, 1969; Wagner 1969a). Combining the asymptotic argument with the

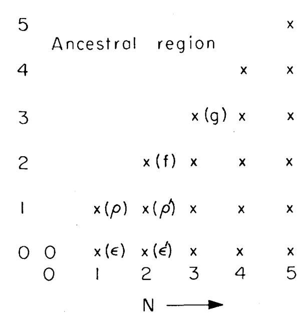

FIG. 3.4. Chew–Frautschi plot showing mass spectrum of $\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y)$. Note the absence of ancestors and also of the possible ghost state, of $(\text{mass})^2=-\frac{1}{2}$, at $L=0$. The first few states are labeled with the names of known mesons. Compare Figure 2.3.

[12] The zeros of the entire function

$$E(x,y)=\Gamma^{-1}(1-x)\Gamma^{-1}(1-y)F(x,y)$$

are what has to be fixed in order to prove that F_0 is unique. Unfortunately, the mathematics of entire functions of several complex variables is difficult and largely unknown. In this connection see J. Korevaar and S. Hellerstein, in *Entire Functions and Related Parts of Analysis*† (1968), Hormander† (1966), Fuks† (1963), and Siegal† (1948). The several complex variable aspect of the problem makes the derivation of rigorous results, including positivity of widths, nearly intractable.

numerical one, it is probably possible to construct a rigorous proof of positivity.

The first few relative values of $H(N, L)$ are shown in Table 3.1. A convenient formula for the $H(N, L)$ may be obtained as follows. We first note[13]

$$T_N(x) \equiv \Gamma(N+x)/\Gamma(x) = [T_N(d/d\mu)e^{\mu x}]_{\mu=0}. \quad (3.24)$$

Using the Gegenbauer expansion,[14] we obtain

$$e^{\mu t} = e^{-\mu s/2} \sum_{L=0}^{\infty} (2L+1) i_L(\mu s/2) P_L(1+2t/s), \quad (3.25)$$

where $i_L(z)$ is the modified spherical Bessel function of the first kind:

$$i_L(z) = \tfrac{1}{2}\pi^{1/2} \sum_{m=0}^{\infty} \frac{(z/2)^{L+2m}}{m!\Gamma(L+\frac{3}{2}+m)}. \quad (3.26)$$

For the Nth pole, we set $s = N - a$, so that

$$H(N, L) = \{[2/\Gamma(N)] T_N(d/d\mu) e^{\mu a} \times \exp[-\tfrac{1}{2}\mu(N-a)] i_L[\tfrac{1}{2}\mu(N-a)]\}_{\mu=0}. \quad (3.27)$$

We can achieve a qualitative understanding of how $H(N, L)$ behaves as a function of N and L by going back to the defining integral, (3.23), and examining the integrand pictorally. The polynomial, $T_N(x)$, is of

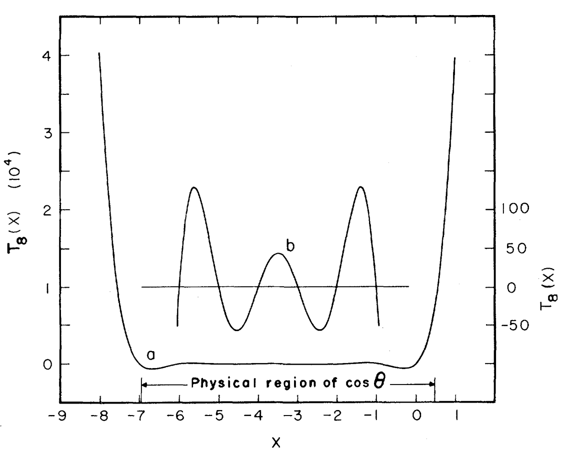

FIG. 3.5. The behavior of $T_8(x)$, the pole residue of the eighth tower in

$$F_0(x, y) = \sum_{K=1}^{\infty} \frac{T_K(x)}{y-K}.$$

The physical region in cos Θ is shown for an intercept slightly less than $\frac{1}{2}$. Curve a, associated with the right-hand scale, shows $T_8(x)$ *in toto*; curve b, associated with the left-hand scale, shows the central oscillations of (T_8x) magnified by 10^4 with respect to curve a.

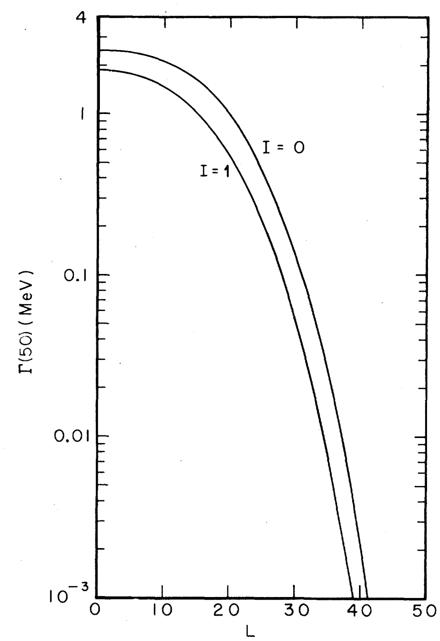

FIG. 3.6. Width of resonances in the 50th tower as a function of their angular momentum L.

Nth order in x, with parity $(-1)^N$ around the symmetry point $x = -\frac{1}{2}(N-1)$, and with oscillations that increase in magnitude as we leave the symmetry point. A picture of $T_8(x)$ is shown in Fig. 3.5.

Clearly, $T_N(x)$ has N integrally spaced zeros, which for $a = \frac{1}{2}$, are spread across the physical region in z in such a way that the ends of the chain are at $z_1 = -1$ and $z_2 = 1 - 2a/(N-a)$. Since the amplitude of the oscillations in $T_N(x)$ increases linearly around $x = -\frac{1}{2}(N-1)$, for $a = \frac{1}{2}$ and large N, in (3.23) we are effectively integrating over the forward peak between $z = z_2$ and $z = 1$.[15] Since $P_L(1) = 1$, the integral is positive. Shapiro (1969) discusses this more fully.

As $a - \frac{1}{2}$ becomes negative, the most backward zero moves into the physical region and some widths become negative. If we use our formula for $H(N, L)$ in terms of i_L, we see that $H(2, 0) = 0$ for $a = \frac{1}{2}$ and this width is the first to go negative as a decreases. In agreement with Frampton and Nambu (1969), asymptotically there are no negative widths created until a reaches $\frac{1}{3}$, at which point $H(N, L) = 0$ for $N - L$ odd. This last remark is easy to verify from the formula (3.27); we

---

[13] The polynomials $T_N(x)$ appear in connection with the spin-matrix polynomials (Nelson, 1969). It turns out that the s wave widths are related to the generalized Bernoulli polynominals. See Nielsen† (1923) and Jordan† (1947). The T_N do not form an orthogonal set, but are related by difference operations.

[14] See Watson† (1966), pp. 128 and 368 ff.

[15] In other words, each pole residue of $F_0(x, y)$ has the typical angular behavior associated with Regge pole exchange. See Eden† (1967), Chap. 9, and Collins and Squires† (1968), VIII.6. The slope of the forward charge exchange peak in $\pi^+\pi^- \to \pi^0\pi^0$ compares favorably with typical experimental values for other processes using a trajectory slope near 1 BeV^{-2} (Shapiro and Yellin, 1968; Shapiro, 1969). This is so far the only phenomenological success of the Veneziano model. (The ratio $\Gamma_f/\Gamma_s = 0.8$ is not a success of this particular model, but results independently from the FESR. See Schmid and Yellin (1969) and also Sec. X below.

have

$$T_N\left(\frac{d}{d\mu}\right)\exp\left(\tfrac{3}{2}\mu a-\tfrac{1}{2}\mu N\right)\underset{a\to 1/3}{\longrightarrow}T_N\left(\frac{d}{d\mu}-\tfrac{1}{2}(N-1)\right)$$

$$=(-1)^N T_N\left(-\frac{d}{d\mu}-\tfrac{1}{2}(N-1)\right) \quad (3.28)$$

and $i_L(z)=(-1)^L i_L(-z)$. For large N and fixed $L\ll(N)^{1/2}$, the N behavior of the residues is approximately

$$H(N,L)\underset{L\ \text{fixed},N\to\infty}{\sim} N^{a-1}/\log N, \quad (3.29)$$

corresponding to the usual Regge asymptotic behavior times logarithmic shrinkage (see Collins and Squires,† 1968, Sec. VIII. 6), while the L behavior for large fixed N and $L\ll(N)^{1/2}$ is

$$H(N,L)\underset{N\ \text{fixed},L\gg 1}{\sim}\exp\left[-(L^2/N)\log N\right]. \quad (3.30)$$

A plot of the widths $H(N,L)$ for $N=50$ is shown in Fig. 3.6, which illustrates the behavior (3.30).

The results above may be verified by using [EIT, 4.14(33, 35)] (compare Blankenbecler and Goldberger, 1962; Kugler, 1968)

$$H(N,L)\cong\int_{1-2a/N}^{1}dzJ_0\left[2L\left(1-\frac{z}{a}\right)^{1/2}\right]\frac{N^y}{\Gamma(y)}$$

$$\cong 2N^a\int_0^\infty dk\exp(-N\log Nx)J_0[2L(x)^{1/2}]$$

$$\times\sum_{j=0}^{\infty}C_j^{(N)}x^j$$

$$\cong 2N^a\sum_{j=0}^{\infty}C_j^{(N)}\Gamma(j+1)(N\log N)^{-j-1}$$

$$\times\exp\left(-\frac{L^2}{N\log N}\right)L_j\left(\frac{L^2}{N\log N}\right), \quad (3.31)$$

where L_j is a Laguerre polynomial and we have used[16] [GR, 8.722(1)]

$$P_L(z)\cong J_0[(2L+1)\tfrac{1}{2}(1-z)^{1/2}] \quad (3.32)$$

for $L\gg 1$ and $1-z\ll 1$.

The behavior in L is therefore that of a model in which there is an impact parameter which grows as $s^{1/2}$ (up to logarithmic factors). The largest $H(N,L)$ occur for $L\lesssim(N)^{1/2}$. Sivers and Yellin (1969b), Drago and Matsuda (1969), and Oehme (1969a) all discuss this behavior. This model has, as one might expect, no absorption in it. Partial waves are roughly constant out to some maximum L, beyond which they fall exponentially.[17]

[16] Strictly speaking, one should verify that (3.31) converges. This can be done by using the results listed in EHF, 10.15 and Buchholz† (1969), Chap. III.

[17] This behavior occurs in any rising trajectory model in which asymptotic behavior is determined by moving J plane poles only.

J. J-Plane Structure

The structure of the model partial wave amplitudes as a function of complex angular momentum is nearly the simplest possible: they have poles in J whose location changes with energy in the $I=0$ and 1 channels, and fixed poles for $I=0$ and 2. Part of this is clear already from the discussion of asymptotic behavior in Sec. III.E. Since $F(x,y)$ has pure moving power behavior x^{y-n} as x gets big and for fixed y, there can be no J-plane cuts. Cuts would induce something more complicated, the usual guess being a logarithmic dependence (Oehme, 1964; Rothe, 1966). In fact, since $F(x,y)$ has no signature in the x or y channels, there can be no right signature fixed poles either, since these would generate fixed power behavior, x^m. This explains the conclusion about the $I=1$ partial wave.

As we have seen, however, $F(a+bs, a+bu)$ has exponentially decreasing behavior for $|\nu|=|\tfrac{1}{2}(s-u)|\to\infty$ at fixed t. This amplitude is even (signatured) in ν and therefore the only possible J-plane singularities are (wrong signature) fixed poles at the "nonsense" points $J=-1, -3, -5, \cdots$, which would not affect the asymptotic behavior. These poles in fact exist and contribute to the $I=0$ and 2 amplitudes.

For illustrative purposes, let us derive the form of the Regge residues and trajectory functions. Using (3.19) and (3.20), we have the expansions

$$A_1^t(\nu,t)=g\sum_{K=1}^{\infty}\frac{\Gamma(K+t+\frac{1}{2})}{\Gamma(K)\Gamma(t+\frac{1}{2})}$$

$$\times\{[-\nu+\tfrac{1}{2}(1-t)-K]^{-1}-[\nu+\tfrac{1}{2}(1-t)-K]^{-1}\}, \quad (3.33a)$$

$$A_2^t(\nu,t)=g\sum_{K=1}^{\infty}(-1)^K\frac{\Gamma(K+t)}{\Gamma(K)\Gamma(t)}$$

$$\times\{[-\nu+\tfrac{1}{2}(1-t)-K]^{-1}+[\nu+\tfrac{1}{2}(1-t)-K]^{-1}\}, \quad (3.33b)$$

which we can think of as fixed-t dispersion relations in ν with a discontinuity equal to a sum of Dirac delta functions. Proceeding in the usual manner (see Eden† 1967, Sec. 5.3), we define the partial wave signatured amplitudes

$$a_{\pm}^I(J,t)=\pi^{-1}\int_0^\infty dzQ_J(z)[D_R^I(t,z)\pm D_L^I(t,z)], \quad (3.34)$$

where $z=\cos\theta_s=\nu/2t$, and where by D_L we mean the left-hand ($\nu<0$) δ-function discontinuity of (3.33). We then get

$$a_+^1(J,t)=0, \quad (3.35a)$$

$$a_-^2(J,t)=0, \quad (3.35b)$$

$$a_-^1(J,t)=2g\sum_{K=1}^{\infty}\frac{\Gamma(K+t+\frac{1}{2})}{\Gamma(K)\Gamma(t+\frac{1}{2})}Q_J\left(1+\frac{2K-1}{t}\right), \quad (3.36a)$$

$$a_+^2(J,t)=2g\sum_{K=1}^{\infty}(-1)^K\frac{\Gamma(K+t)}{\Gamma(K)\Gamma(t)}Q_J\left(1+\frac{2K-1}{t}\right). \quad (3.36b)$$

We are looking for the singularities of $a_-^1(J,t)$. Each term in (3.36a) has the fixed poles at negative integral J present in the Legendre function, $Q_J(z)$, whose analytic properties in J are evident from the relation [EHF, 3.2(5)]

$$Q_J(z)=[\pi^{1/2}\Gamma(J+1)/\Gamma(J+\tfrac{3}{2})(2z)^{J+1}]\times{}_2F_1[\tfrac{1}{2}J+1,\tfrac{1}{2}J+\tfrac{1}{2};J+\tfrac{3}{2};(1/z^2)]. \quad (3.37)$$

To see whether they are present in the partial wave amplitude, we must use the fact that the residue of the pole at $J=-N$ of $Q_J(z)$ is $P_{N-1}(z)$ to compute

$$\gamma_1^N(t)=2g\sum_{K=1}^{\infty}\frac{\Gamma(K+t+\frac{1}{2})}{\Gamma(K)\Gamma(t+\frac{1}{2})}\times P_{N-1}\left(1+\frac{2K-1}{t}\right)\quad(t+\tfrac{1}{2}<-N), \quad (3.38a)$$

$$\gamma_2^N(t)=2g\sum_{K=1}^{\infty}\frac{(-1)^K\Gamma(K+t)}{\Gamma(K)\Gamma(t)}\times P_{N-1}\left(1+\frac{2K-1}{t}\right)\quad(t<-N). \quad (3.38b)$$

In fact, $\gamma_1^N(t)=0$. This is most easily shown for the case $N=1$, where (3.38a) becomes

$$\gamma_1^1(t)=2g\sum_{K=1}^{\infty}\frac{\Gamma(K+t+\frac{1}{2})}{\Gamma(K)\Gamma(t+\frac{1}{2})}\qquad(t+\tfrac{1}{2}<-1), \quad (3.39)$$

which vanishes because of

$$\lim_{P\to\infty}\sum_{K=1}^{P}\frac{\Gamma(K+t+\frac{1}{2})}{\Gamma(K)\Gamma(t+\frac{1}{2})}=\frac{\Gamma(P+1+t+\frac{1}{2})}{\Gamma(P)\Gamma(t+\frac{1}{2})(t+\frac{3}{2})}\to 0\qquad(t<-\tfrac{1}{2}). \quad (3.40)$$

This is in agreement with our comments above. That the residues of the other fixed poles vanish can be shown in a similar manner. The situation is different for the case of the $I=2$ fixed poles where $\gamma_2^1(t)$ does not vanish. The residue of the first $I=2$ fixed pole is

$$\begin{aligned}\gamma_2^1(t)&=2g\frac{\sin\pi t}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(-t)\Gamma(m+1+t)}{\Gamma(m+1)}\\&=2g\frac{\sin\pi t}{\pi}\int_0^1 du\,u^t(1-u)^{-t-1}(1+u)^{-1}\\&=-gt2^{-t}.\end{aligned} \quad (3.41)$$

The other residues of the fixed poles at wrong-signature (odd) negative integers can also be shown to be nonzero (Fivel and Mitter, 1969; Drago and Matsuda, 1969; Sivers and Yellin, 1969a; Allesandrini and Amati, 1969).

We now examine (3.36a) and (3.36b) for moving singularities which appear as divergences of the infinite sums. Using the asymptotic expansion (3.15) for the gamma function and the large z expansion of (3.37), we find

$$a_1^-(J,t)\sim g\pi^{1/2}\left(\frac{t}{4}\right)^J\frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})\Gamma(t+\frac{1}{2})}\times\sum_{K=1}^{\infty}[K^{t+\frac{1}{2}-J-1}+O(K^{t+\frac{1}{2}-J-2})], \quad (3.42)$$

which has a simple pole at $J=t+\frac{1}{2}=\alpha(t)$, as can be seen from the analytic structure of the Riemann zeta function:

$$\zeta(z)=\sum_{K=1}^{\infty}K^{-z}=(z-1)^{-1}+E_\zeta(z). \quad (3.43)$$

The residue of the leading Regge (moving) pole is then[18]

$$\beta_0(t)=g\{\pi^{1/2}\alpha(t)/\Gamma[\alpha(t)+\tfrac{3}{2}]\}(\tfrac{1}{4}t)^{\alpha(t)}, \quad (3.44)$$

where $(t/4)^\alpha=q^{2\alpha}$ is the usual threshold factor. Removing the leading divergence from (3.42), we find nonleading Regge poles (satellites) with residues

$$\beta_1(t)=\tfrac{1}{8}(g\pi^{1/2})[\alpha(q^2)^{\alpha-1}/\Gamma(\alpha+\tfrac{1}{2})], \quad (3.45)$$

$$\beta_2(t)=(g\pi^{1/2}/96)[\alpha(\alpha-2)(q^2)^{\alpha-2}/\Gamma(\alpha-\tfrac{1}{2})], \quad (3.46)$$

$$\beta_3(t)=(g\pi^{1/2}/384)\alpha[\alpha^2+3-\tfrac{3}{2}\alpha][(q^2)^{\alpha-3}/\Gamma(\alpha-\tfrac{1}{2})]. \quad (3.47)$$

In (3.44) the factor $\Gamma^{-1}(\alpha+\frac{3}{2})$ appears because of the Mandelstam (1962) symmetry of the partial wave amplitudes

$$a(-J-\tfrac{1}{2},t)=a(J-\tfrac{1}{2},t)\qquad(J\text{ integral}). \quad (3.48)$$

[18] Fivel and Mitter (1969) give the following expression for the $\beta_n(t)$, using the approach of Khuri (1968):

$$\beta_n(\alpha)=\frac{g\pi^{1/2}(-1)^n\alpha q^{2\alpha-2n}}{2^{n+1}\Gamma(\alpha+3/2-n)}\sum_{p=0}^{[n/2]}\frac{(t/2)^{2p}\xi_{n-2p}(\alpha)}{\Gamma(n-2p+1)}\cdot\frac{\Gamma(\alpha+3/2-n)}{\Gamma(\alpha+3/2-n+p)},$$

where $[n/2]$ is the integer part of $n/2$ and

$$\xi_j(\alpha)=(d/dz)^j[e^{-z/2}(\sinh z/z)^{-\alpha-1}]_{z=0}.$$

Note that the trajectories begin compensating each other as we reach the level of the third daughter instead of having zeros at all half-integers. [If no compensation occurred, $\beta_n(\alpha)$ would contain the factor $\Gamma(\alpha+\frac{3}{2}-n)$.]

We can check that the $I=2$ amplitude has no moving poles by noting [GR, 9.522(2)] that

$$\sum_{K=1}^{\infty}(-1)^K K^{-z}=(2^{1-z}-1)\zeta(z). \tag{3.49}$$

Looking at the form of the $I=0$ amplitude, we see that we can write it as a linear combination of $A_1^-(\nu, t)$ and $A_2^+(\nu, t)$:

$$a_0^+(J, t)=\tfrac{3}{2}a_1^-(J, t)-3a_2^+(J, t), \tag{3.50}$$

so that we have both fixed poles and moving poles of isospin 0, again in agreement with our remarks above.

K. Exchange Degeneracy

From (3.50) we see that the $I=0$ Regge trajectory, the f trajectory, is degenerate with the $I=1$ rho trajectory. This is a general feature of narrow resonance models which have no resonances in a particular channel. In this case, exchange degeneracy (Arnold, 1965) is guaranteed by the absence of resonances in the physical t region for the $I=2$ amplitude.

L. Sum Rules at $t=0$

If we use the formula (3.33a) for the $I=1$ amplitude at $t=0$, we get the $\pi\pi$ sum rule for this model (Yellin, 1969b):

$$\pi=\sum_{K=1}^{\infty}\frac{\Gamma(K-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(K)(K-\frac{1}{2})}=2+\frac{1}{3}+\frac{3}{20}+\frac{5}{56}+\cdots. \tag{3.51}$$

In (3.51), the contribution of the (ρ, ϵ) tower is 2, the f tower yields $\frac{1}{3}$, etc. (The states are named in Fig. 3.4.) Curiously enough, these contributions are in qualitative agreement with the phenomenological estimate of Gilman and Harari (1968).

Along $t=0$, A_2^t vanishes. Explicitly in (3.33b) we see that the factor $\Gamma^{-1}(t)$ accounts for this. This means that the contribution of each tower to the discontinuity in ν along $t=0$ is zero.

If we write out the $I=2$ discontinuity in ν for arbitrary t, up to an inessential overall factor we have

$$\begin{aligned}D_2(\nu, t)=&\sum_{K=1}^{\infty}(-1)^K\frac{\Gamma(K+t)}{\Gamma(K)\Gamma(t)}\\&\times\{\delta[\nu+\tfrac{1}{2}(1-t)-K]-\delta[-\nu+\tfrac{1}{2}(1-t)-K]\}\\&=\tfrac{1}{4}[P_0(z_s)-P_1(z_s)]\delta(\nu-\tfrac{1}{2})\\&+\tfrac{3}{8}[P_2(z_s)-P_1(z_s)]\delta(\nu-\tfrac{3}{2})+\cdots+\{z_s\to z_u, \nu\to-\nu\}.\end{aligned} \tag{3.52}$$

We see here, for example, that ρ and ϵ cancel each other at $z=-1$, $t=0$. If we translate this into a statement about the widths, putting in the $(2J+1)$ factors and the isospin Clebsch–Gordan coefficients, we get

$$\Gamma_\rho/\Gamma_\epsilon=\tfrac{1}{3}\times\tfrac{2}{3}=\tfrac{2}{9}, \tag{3.53}$$

which is the same result one gets from the current algebra sum rules (Gilman and Harari, 1968). (See the discussion in Sec. V for more about current algebra and the $\pi\pi$ narrow resonance amplitude.)

M. Finite Energy Sum Rules

The usual $I=1$ and 2, finite energy sum rules (FESR's) for a Regge-behaved amplitude of the type we are considering, for arbitrary t, are

$$\frac{1}{2}\int_{-N}^{+N}\nu\, d\nu D_2^t(\nu, t)=0, \tag{3.54}$$

$$\frac{1}{2}\int_{-N}^{+N}d\nu D_1^t(\nu, t)=\tilde{\beta}\frac{N^{\alpha+1}}{\alpha+1}+\cdots, \tag{3.55}$$

where $\tilde{\beta}$ is the Regge residue with the $q^{2\alpha}$ factor removed (reduced residue). Because even and odd spins have opposite signs in (3.52), we can expect an oscillating behavior with the amplitude of the oscillations increasing with t, for $t>0$. Choosing N such that $-\frac{1}{2}+K+\frac{1}{2}t\leq N\leq-\frac{1}{2}+K+\frac{1}{2}t+1$, the highest tower included has [cf. (3.33a) and (3.33b)] index K, and by induction we get

$$\frac{1}{2}\int_{-N}^{+N}\nu\, d\nu D_2^t(\nu, t)=\tfrac{1}{2}(-1)^N(N+t)\frac{\Gamma(t+N)}{\Gamma(t)(N)}, \tag{3.56}$$

$$\begin{aligned}\frac{1}{2}\int_{-N}^{+N}d\nu D_1^t(\nu, t)&=\frac{\Gamma(N+1+\alpha)}{\Gamma(N)\Gamma(\alpha)(\alpha+1)}\\&\sim\frac{N^{\alpha+1}}{(\alpha+1)\Gamma(\alpha)}\left[1+\frac{(\alpha+1)\alpha}{2N}+O(N^{-2})\right].\end{aligned} \tag{3.57}$$

As expected, (3.56) oscillates as each succeeding tower is added, while (3.57) yields a sum over the contributions of the Regge trajectories.[19]

N. Uniqueness

At this point we would like to speculate on the possibility that Eq. (3.13) is a unique solution to the narrow resonance $\pi\pi$ amplitude under the assumptions (a)–(f) of Sec. III.C.[20] We have not constructed a

---

[19] The peculiar form of the pole residues in $F_0(x, y)$ leads to a special feature of the model which makes possible an "exact" bootstrap. The point is that $\Gamma(N+Y)/\Gamma(N)\Gamma(Y)$ is not only a polynomial of order N in Y, but if y is integral, it is a polynomial of order y in N. This means that, for $\alpha(t)=$integer, a finite number of Regge trajectories contribute to the right-hand side of the FESR, and we can calculate the partial $\pi\pi$ widths by successive iterations of (3.57), using the various moments ν^n. This gives a possible uniqueness proof very similar to the construction of Schmid (1969a).

[20] Much of the argument of this section has been developed by one of us (J.Y.) in collaboration with R. F. Dashen.

proof of this conjecture, but it is interesting to try to find a counterexample in order to see how the various assumptions constrain the model.

Except for assumption (a), the narrow resonance approximation, all the assumptions listed are physical ones. In Sec. VI we will further examine amplitudes which contain cuts and violate assumption (a). There we will see that it is possible to remove the poles from the real axis, keeping all other desirable properties. This has been done by Suzuki (1969). In this procedure the input widths are unconstrained.

As for the analyticity requirement, assumption (c), we can argue that the polynomial residue, $G(K, y)$, must be of order K because of crossing and Regge behavior. That is, as $y \to \infty$, for fixed x, we must have

$$\lim_{y\to\infty} [G(K, y)/(x-K)] = y^K/(x-K) \qquad (3.58)$$

or else we will get the wrong set of Regge trajectories.

Requirement (e) eliminates representations with poles in three channels, as we will discuss in Sec. IV. If we relax requirement (e) and attempt to break exchange degeneracy (Mandelstam, 1968b), the positivity condition seems to be violated.

Condition (b), Regge behavior, is certainly necessary. If we did not require Regge behavior, we could have functions of the form

$$F_0(a+bs, a+bu)+F_0[a+(bs/2), a+(bu/2)], \qquad (3.59)$$

where the second term has all the same properties as the first but leaves out some of the particles in the spectrum. Conversely, we see that if we try to vary the spacing of poles and zeros in the model, the asymptotic behavior is no longer Regge-behaved (Bali, Coon, and Dash, 1969b).

The positivity condition, assumption (d), prohibits the use of subsidiary terms like

$$\frac{\Gamma(K-x)\Gamma(P-y)}{\Gamma(K+P+M-x-y)}+\frac{\Gamma(K-y)\Gamma(P-x)}{\Gamma(K+P+M-x-y)}, \qquad (3.60)$$

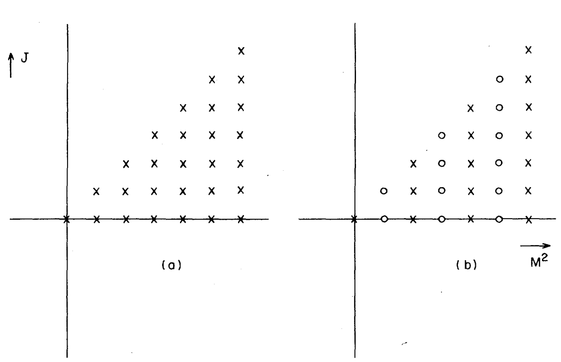

FIG. 4.1. Chew–Frautschi plot for mass spectrum of (a) Veneziano representation and (b) Virasoro representation.

which give an infinite number of negative widths[21] (Shapiro, 1969).

Requirement (f) fills a trivial hole in the positivity requirement. Referring again to Fig. 3.3, we see that all the zeros of F_0, except the one mentioned in (f), play the role of preventing double poles from occurring at the intersection of the x and y poles in the double spectral region, $(x, y)>0$. Requirement (e) tells us that, except for the PCAC zero at $t=0$, the other zeros are straight lines in the Re x–Re y plane. If we move the PCAC zero by an infinitesimal amount, then all widths change only infinitesimally, and we could generate a counterexample,

$$\frac{t+\epsilon}{t}F_0(x, y)=\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}+\epsilon\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(2-x-y)}, \qquad (3.61)$$

where we must choose the sign of ϵ so that $H(2, 0)$ is positive since for $a=\frac{1}{2}$ and $\epsilon=0$, $H(2, 0)=0$.

If we try to shift the zeros around more drastically,

[21] In connection with the positivity requirement, there arises the following problem: What restrictions are there on the positions of the roots $\{x_i^N\}$ of the polynomial $R_N(x)$ of order N, such that all its Legendre coefficients, λ_k^N, $R_N(x)=\sum_{k=0}^N \lambda_k^N P_k(x)$, are ≥ 0? A not very useful constraint on the $\{x_i^N\}$ is that the sufficient and necessary condition

$$\frac{\det\begin{vmatrix} y_0 & y_1 & \cdots & y_N \\ P_0(x_1^N) & \cdots & \cdots & P_N(x_1^N) \\ \vdots & & \cdots & \vdots \\ P_0(x_N^N) & \cdots & \cdots & P_N(x_N^N) \end{vmatrix}}{\det\begin{vmatrix} 1 & 1 & \cdots & 1 \\ P_0(x_1^N) & \cdots & \cdots & P_N(x_1^N) \\ \vdots & & \cdots & \vdots \\ P_0(x_N^N) & \cdots & \cdots & P_N(x_N^N) \end{vmatrix}} \geq 0$$

hold for all $y_i \geq 0$, provided the $\{x_i^N\}$ are distinct. Information about this problem can be found in Szëgo† (1939), Akhiezer and Krein† (1938), and especially Marden† (1949).

positivity is destroyed.[12] For example, consider

$$\left[1+\epsilon\left(1+\frac{xy}{t}\right)\right]F_0(x,y)$$
$$=\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}+\epsilon\frac{\Gamma(2-x)\Gamma(2-y)}{\Gamma(2-x-y)},\quad(3.62)$$

where the residue of the pole at $x=N$ is now

$$y(y+1)\cdots(y+N-1)\to y(y+1)\cdots(y+N-1)+\epsilon Ny^2(y+1)\cdots(y+N-2).\quad(3.63)$$

No matter how small ϵ is chosen, there will always be a range of N for which $N\epsilon$ is large and the second term in (3.63) dominates the first term. The sign of the contribution of the second term to the width of a spin L state relative to the first is

$$(-1)^{N+1+L}\,\mathrm{sg}\,(\epsilon),\quad(3.64)$$

so that there will be an infinite number of negative widths either for $(N+L)$ odd or for $(N+L)$ even, depending on the sign of ϵ.[22] A similar thing happens when we consider the example of Matsuda (1969b):

$$s(x,y;\lambda)=\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}\,{}_2F_1[-x,-y;\tfrac{1}{2}(1-x-y);\lambda].\quad(3.65)$$

No matter how small λ is in this expression, we conjecture that there are an infinite number of negative widths.

Additional material relevant to this section can be found in Atkinson and Dietz (1969), Boguta (1969), Moffat (1969), and Tokuda (1969).

IV. MODIFICATIONS OF THE FOUR-POINT FUNCTION

Most work on narrow resonance models has made explicit use of the form

$$C(s,t)=\sum_{m,n,p}\frac{\Gamma(m-x)\Gamma(n-y)}{\Gamma(m+n+p-x-y)}c_{mnp}\quad(4.1)$$

with a small number of terms in the sum. This is usually done for simplicity because the properties of the beta function are relatively well known. There are, however, other possibilities for the functional form of a narrow resonance model which we will discuss in this section.

A. Virasoro's Representation

An alternative to Veneziano's beta function form was proposed by Virasoro (1969a). Although Virasoro's model can be formulated for other amplitudes (Hara, 1969; Virasoro, 1969a), it takes its simplest form for a reaction such as $\pi\pi\to\pi\omega$, where the factored amplitude is completely symmetric in s, t, and u. Let $x=a+bs$, $y=a+bt$, and $z=a+bu$. Virasoro suggested using

$$V(s,t,u)=\frac{\Gamma(-x/2)\Gamma(-y/2)\Gamma(-z/2)}{\Gamma(-(x+y)/2)\Gamma(-(x+z)/2)\Gamma(-(y+z)/2)}.\quad(4.2)$$

This form simultaneously exhibits the poles in all three channels and has definite signature trajectories only. The spectrum of poles in Virasoro's model is compared with that of Veneziano's in Fig. 4.1. The partial wave projection, $a(J,x)$, of Virasoro's amplitude has *multiplicative* fixed poles which appear in the Regge residues at negative wrong signature integers:

$$\beta_{\text{Virasoro}}(x)=E(x)q^{2x}\frac{\Gamma(x/2+1/2)}{\Gamma(x/2-D/2)\Gamma(x+3/2)},\quad(4.3)$$

where $D=x+y+z=3a+b(\sum_i m_i^2)$ and $E(x)$ is entire. These poles therefore do not affect the asymptotic behavior of the physical amplitudes, but appear only in the asymptotic expansion of the signatured amplitudes. In the Veneziano form, as discussed in Sec. III, a term like $B(-y,-z)$ contributes *additive* fixed poles to the partial wave projection, $a(J,x)$, also at the negative wrong signature integers. The fixed poles in both the Virasoro and Veneziano representations seem to be manifestations of the Gribov–Pomeranchuk phenomenon (Gribov and Pomeranchuk, 1962) and violate unitarity.[23] A more complete theory would be expected to have cuts in the complex J-plane to shield these poles. Once cuts are allowed, there seems to be no way of eliminating one representation in favor of the other on the basis of the properties of its fixed poles.

One property of Virasoro's representation which sometimes proves inconvenient is the presence of poles in all three channels. Suppose, for example, that one wishes to use the Virasoro representation for $\pi\pi$ scattering. Although the intercept of the leading trajectory in the $\pi^+\pi^+$ channel can be made as negative as we like, we cannot eliminate the exotic poles in this channel entirely (Virasoro, 1969a). For reactions in which one channel has exotic quantum numbers, the assumed absence of poles in this channel can be used to restrict the number of Veneziano terms considered, and leads

[22] This bad asymptotic behavior of widths comes from altering the fixed angle behavior of $F_0(x,y)$. We thank S. Mandelstam for pointing out to us the importance of fixed angle behavior in the uniqueness problem.

[23] The argument that the wrong signature fixed poles of narrow resonance models are manifestations of the Gribov–Pomeranchuk phenomenon, and do not represent the potential scattering poles which begin to move as a coupling is turned on, is due to Mandelstam (1969a). As shown in Sec. III, the additive fixed poles in Veneziano's representation have energy-dependent residues behaving as 2^{-t}. According to Mandelstam, the essential singularity at $t=-\infty$ replaces the usual left-hand cut, and $F_0(\alpha(s),\alpha(u))$, as far as t-channel effects, will be similar to the third double-spectral function. If this is correct, three-particle intermediate states will produce cuts in the J-plane, in the lowest order in which they appear, for any reasonable unitarization scheme.

immediately to exchange degeneracy. For example, in Sec. III we saw how eliminating exotic $\pi^+\pi^+$ resonances gives degeneracy between the ρ and f trajectories. Since, in Virasoro's representation, all channels contain poles, one cannot have this simple exchange degeneracy. This is one reason why Virasoro's model was eliminated from consideration in Sec. III.N, by requiring that there be no $I=2$ poles.

Mandelstam (1969a) has found an integral representation of Virasoro's amplitude. Virasoro (1969c) and Collop (1970) have generalized Virasoro's model to N particles. So far as we know, there has been no thorough examination of the factorization or positivity properties of the model.[24]

B. The Generalization of Mandelstam

Mandelstam (1969a) has found an integral representation for a narrow resonance model which has linear trajectories, polynomial residues, crossing symmetry, Regge behavior, and which includes the Veneziano and Virasoro representations as special cases. The formula is

$$M(x,y,z)=\iint_R d\lambda\, d\eta\lambda^{-x-2}\eta^{-y-2}(2-\lambda-\eta)^{-z-2}$$
$$\times\left\{\frac{1-\lambda}{\eta(2-\lambda-\eta)}\right\}^{\nu_1}\left\{\frac{1-\eta}{\lambda(2-\lambda-\eta)}\right\}^{\nu_2}\left\{\frac{\eta+\lambda-1}{\lambda\eta}\right\}^{\nu_3}, \quad (4.4)$$

where the ν_i are arbitrary and the range of integration, R, is the triangle

$$\eta<1, \lambda<1, \quad \text{and} \quad \lambda+\eta>1, \quad (4.5)$$

shown in Fig. 4.2. The formula (4.4) is a special case of

$$\tilde{M}(x,y,z)=\int_R d\lambda\, d\eta\lambda^{-x-2}\eta^{-y-2}(2-\lambda-\eta)^{-z-2}F(\lambda,\eta), \quad (4.6)$$

where $F(\lambda,\eta)$ is assumed analytic in R with the exception of possible power branch points along the boundaries of R. Expanding $F(\lambda,\eta)$ in a power series, we can express (4.6) as a sum of terms of the form (4.4).

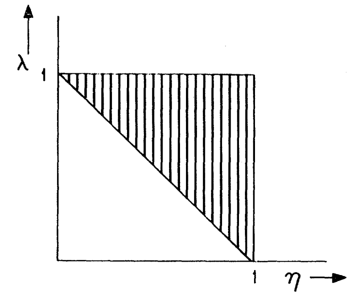

FIG. 4.2. The region of integration for the integral representation of Mandelstam's generalization of the Veneziano and Virasoro representations.

The integral representation for the Virasoro amplitudes (4.2) is a special case of (4.4), with

$$\nu_1=\nu_2=\nu_3=-1/2(D+3). \quad (4.7)$$

The Veneziano form appears if one or more of the ν_i are -1. The integral then diverges along one side of the triangle and we find

$$\lim_{\nu_3\to-1}(\nu_3+1)M(x,y,z)=\int_R d\lambda\lambda^{-x-1}\eta^{-y-1}\, d\eta\delta(\eta+\lambda-1)$$
$$=\int_0^1 d\lambda\lambda^{-x-1}(1-\lambda)^{-y-1}, \quad (4.8)$$

which is the familiar integral representation of the beta function [EHF, 1.5, (1, 2)]. For more details of the properties of this amplitude, the reader is referred to the Mandelstam paper quoted above. So far, Mandelstam's amplitude has not been extended to N particles, and the factorization and positivity properties of its residues have not been thoroughly examined.

C. Altering the Resonance Structure

Suppose we ask in what way the patterns of resonances shown in Fig. 4.1 can be modified. Mandelstam (1969a) has given a partial answer to this question.

Consider the function

$$C(x,y)=\int_0^1 du u^{-x-1}(1-u)^{-y-1}[1-u(1-u)]^{\delta}, \quad (4.9)$$

where δ is an arbitrary constant. For $\delta=0$ this just reduces to a beta function. The formula (4.9) can be considered as a special case of

$$\tilde{C}(x,y)=\int_0^1 du u^{-x-1}(1-u)^{-y-1}f(u), \quad (4.10)$$

where $f(u)$ is analytic in the interval [0, 1] except for possible power branch points at 0 or 1. Clearly $C(x,y)$ can also be written as a series of the form (4.1), with $p=0$, by expanding the integrand in a power series in $u(1-u)$.

[24] In this connection see the discussion of Bitar (1969a), who discusses the Lorentz-pole content of narrow resonance models in a rather transparent manner. If one is to have a narrow resonance amplitude containing one Lorentz pole only, its pole residues must be proportional to $U_N(1-s/2m^2)$ for equal mass scattering, where the Chebiytcheff polynomial [EHF, 10.11(6)] is given by $U_N(\cos q)=\sin[(N+1)q]/\sin q$. Any finite number of residues in the $P_j(z)$ expansion of U_N

$$U_N(1-s/2m^2)=\sum_{j=0}^{N}[d_{0j0}{}^{0,N+1}(\delta)]^2P_j(z_t)$$
$$\equiv\sum_{j=0}^{N}d_j{}^NP_j,$$

where $\cosh\delta=t^{1/2}/2m$, and where in the expansion $t=(N-a)/b$, can be matched with those of the usual Veneziano sum over terms like $\Gamma\Gamma/\Gamma$. The functions $d^{jk}{}_{stu}$ are defined in Appendix C of Bitav and Tindle (1968). Bitar points out that if one tries to match the whole infinity of coefficients, $d_j{}^N$, the resulting $\Gamma\Gamma/\Gamma$ sum no longer is atonous dual. The reader is referred to his paper for further details. See also Paciello *et al.* (1969a) for arguments in a related context.

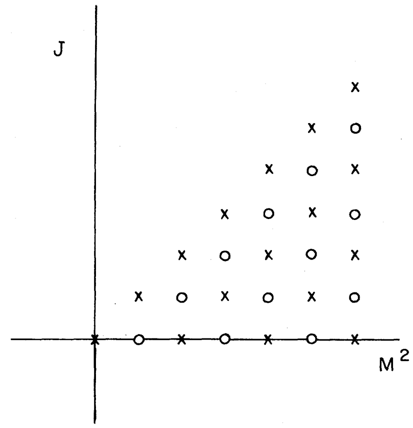

FIG. 4.3. Mass spectrum for vanishing odd satellite trajectories, as in (4.9) and (4.12).

By choosing δ in (4.9) properly, we can get various patterns of resonances without introducing poles into the third channel.[25] For example,

$$\delta = D+1 \tag{4.11}$$

can be shown to yield the Virasoro pattern, Fig. 4.1(b), while

$$\delta = 1/2(D+1) \tag{4.12}$$

makes alternate trajectories vanish as in Fig. 4.3. In fact, using (4.12), we can write (4.9) in the form

$$\begin{aligned} C(x, y)\,|_{\delta=1/2(1+D)} &= B(-x, -y)\,{}_3F_2(-x, -y, -1/2(D+1); \\ &\qquad -1/2(x+y), -1/2(x+y-1); 1/4) \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{1/2(D+1)}{n} B(-x+n, -y+n). \end{aligned} \tag{4.13}$$

It is probably not possible to eliminate any more of the nonleading trajectories than is done in Fig. 4.3. In particular, although we have not been able to prove this conjecture, it does not seem possible to produce patterns such as are shown in Fig. 4.4, where alternate *and* odd signature trajectories have been eliminated. {Such a pattern can be constructed if one is willing to accept an antisymmetric function with poles in all three channels. The prototype example is that of Odorico (1971b):

$$F_0(x, y) \sin[\tfrac{1}{2}\pi(x-y)]/\sin[\tfrac{1}{2}\pi(x+y)]\}.$$

[25] There is a possible confusion in the paper of Mandelstam (1969a). The term "Veneziano formula" used there applies not only to a single $\Gamma\Gamma/\Gamma$ type term, but also to any convergent sum. Without this clarification it might appear that alternate and/or odd signature trajectories can be eliminated only at the expense of breaking exchange degeneracy and introducing exotic resonances. We thank S. Mandelstam for clarifying this point for us.

Strictly speaking, of course, it is not possible to have Regge trajectories of definite signature unless we have singularities in all three channels, so that when we speak of the signature of t-channel resonances in (4.9), we are considering their contribution to $C(x, y) \pm C(z, y)$. Note that an elimination of any one of the n = even trajectories, except $n=0$, from the resonance structure of Fig. 4.3 would violate analyticity just as the elimination of a Freedman–Wang daughter also violates analyticity (Freedman and Wang, 1967; Paciello *et al.*, 1969a; di Vecchia *et al.*, 1969; Scheck, 1969).

Finally, we recall that when dealing with sums of the form (4.1), one can have the difficulty mentioned in Sec. II, in that the asymptotic form may not extrapolate smoothly into the low-energy region. For example, in Eq. (4.9),

$$\lim_{x\to-\infty} C(x, y) = (-x)^y \Gamma(-y), \tag{4.14}$$

which is the same asymptotic form as a beta function, $\Gamma(-x)\Gamma(-y)/\Gamma(-x-y)$, although the resonance structure is quite different. The function $C(x, y)$ therefore does not satisfy an FESR of the form (2.2) for a low value of the cutoff, N.

See also Argyres and Lam (1969), Balachandran (1969), and Bitar (1969b).

V. CURRENT ALGEBRA AND PCAC IN THE NARROW RESONANCE MODEL

In this section we will discuss current algebra and PCAC in terms of the narrow resonance model. Early work along this line was reviewed by Weinberg* (1969). For basic information about current algebra, see Adler and Dashen† (1968).

We will not attempt a complete discussion of all the papers which have appeared on this topic, but will instead consider the specific examples of $\pi\pi$ and πN elastic scattering. We will ask the extent to which

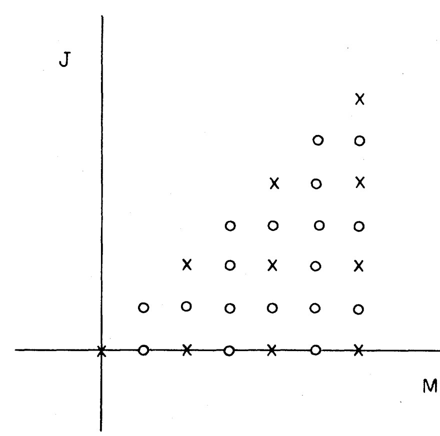

FIG. 4.4. Mass spectrum in which odd satellites and alternate towers vanish. A narrow resonance model with this spectrum probably violates crossing symmetry.

simple narrow resonance models for these amplitudes can be made consistent with the predictions of current algebra and of broken $SU(2)\otimes SU(2)$ symmetry in the form of PCAC.

As we shall see, the more we require of our narrow resonance amplitudes, the more diseases appear, and this again indicates we are not dealing with a fundamentally sound description of reality.

We make an operational distinction between current algebra and PCAC. By "current algebra results," we mean relations holding in the limit of exact $SU(2)\otimes SU(2)$ symmetry, while by "PCAC results" we mean relations which are model dependent in that they depend on assuming a particular form for the chiral symmetry breaking interaction. This distinction will be explained more completely as we examine $\pi\pi$ scattering. Consistency of narrow resonance hadronic amplitudes with PCAC and current algebra has been studied by Lovelace (1968), Kawarabayashi *et al.* (1968), Ademollo *et al.* (1969), and Yellin (1969a, b).

In connection with current algebra and PCAC, the important question arises of whether or not weak and electromagnetic form factors of the hadrons can be determined, even in principle, from a narrow resonance model.[26] Our answer is that the behavior of a hadronic form factor, $F(q^2)$, with respect to its argument, depends on those aspects of the narrow resonance model least likely to be reliable: factorization and nonleading trajectories. As we shall explain below, this puts us in disagreement with those workers who have, for example, derived a form for the pion electromagnetic form factor from simple Veneziano models.

More precisely, we believe that narrow resonance amplitudes do not provide a definitive recipe for making an off-shell continuation leading to an exact form, with q^2 dependence, for example, for the symmetry breaking σ vertices.

A. $\pi\pi$ Scattering and PCAC

Following the arguments of Dashen and Weinstein (1969a), if we insist on a theory in which broken $SU(2)\otimes SU(2)$ symmetry is relevant, the $\pi\pi$ scattering amplitude can be written as

$$T(p_i)\cong\epsilon A+B_0\xi^2+\cdots. \qquad (5.1)$$

In (5.1), isospin indices are suppressed, ϵ is a small parameter measuring the strength of the $SU(2)\otimes SU(2)$ symmetry breaking which is zero if the Goldberger–Treiman relation (Goldberger and Treiman, 1958) is exact, and ξ is a scaling factor for four-momenta such that $p_i=\xi P_i$ for some fixed P_i. The constant B_0 is universal in that it appears in any process $\pi H\rightarrow\pi H$ as the non-Born contribution to the derivative of the crossing odd piece of the amplitude, evaluated at threshold for zero mass pions. In the case of $\pi\pi$ scattering where there is no Born term,

$$B_0=d/d\nu A_1^t(\nu,t)\,|_{s=t=u=\nu=0}\equiv 1/8\pi f_\pi^2, \qquad (5.2)$$

where, as in Sec. III, $\nu=\frac{1}{2}(s-u)$ and f_π is the pion decay constant. This is the current algebra constraint for $\pi\pi$ scattering according to the distinction we made above.

If we assume that $SU(2)\otimes SU(2)$ symmetry breaking proceeds via the $(\frac{1}{2},\frac{1}{2})$ representation, we get

$$\epsilon A=m_\pi^2/8\pi f_\pi^2=m_\pi^2B_0, \qquad (5.3)$$

which leads to Weinberg's scattering length ratio $a_0/a_2=-7/2$ (Weinberg, 1966).

These conclusions are really very general; if the on-shell $\pi\pi$ amplitude is expanded as a power series in the Mandelstam variables about the point $s=t=u=0$ and if $SU(2)\otimes SU(2)$ symmetry is enforced, then current algebra implies that the constant term is zero and the coefficient of the linear term is a universal constant. Introducing a nonzero pion mass breaks the $SU(2)\otimes SU(2)$ symmetry and gives a finite value to the constant term in the power series expansion. The value of the constant depends on the particular choice of the model for symmetry breaking. For example, in the σ model (Gell-Mann and Levy, 1960; Dashen, 1969), the symmetry-breaking piece of the Hamiltonian transforms like $(\frac{1}{2},\frac{1}{2})$ under $SU(2)\otimes SU(2)$ and gives the PCAC prediction $a_0/a_2=-7/2$ (Weinberg 1966; Khuri 1966).

Now suppose we want to make the narrow resonance model discussed in Sec. III consistent with $SU(2)\otimes SU(2)$ symmetry. We first take $m_\pi=0$ and define the system by

$$\begin{aligned}A_2^t(s,u)&=gF_0(x,y)\\&=g[\Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y)],\end{aligned} \qquad (5.4)$$

where the notation is that of Eq. (3.13) and the complete set of t-channel isospin amplitudes is given in terms of gF_0 by Eq. (3.8). To make (5.4) consistent with (5.1) when $m_\pi^2=0$ (and $\epsilon=0$), we want the amplitudes X_I^t to vanish linearly in s, u, and t at the point $s=u=t=0$. By examining (3.8), we see that the way to accomplish this is to insist that $a=\frac{1}{2}$, where a is the rho trajectory intercept, and $x\equiv\alpha_\rho(s)=a+bs$. In (5.4) this gives $1-x-y=1-2a-b(s+u)=bt$, so that the amplitude $A_2^t(s,u)$ vanishes along $t=0$. Of course, this result, $\alpha_\rho(0)=\frac{1}{2}$, depends on the choice of amplitude (5.4) and is therefore not independent of the uniqueness difficulties of the original amplitude discussed in Sec. III. With the particular choice of amplitude (5.4), the energy scale b is related to the universal constant of Dashen and Weinstein by

$$(B_0)_{\text{model}}\equiv g\pi b. \qquad (5.5)$$

The next job is to introduce symmetry breaking. We will choose to do this by letting the intercept, a, vary.

[26] In connection with the self-consistent determination of form factors, see Dashen and Frautschi (1966a, b) and S. Mandelstam (1966).

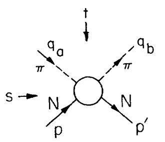

FIG. 5.1. πN scattering.

Letting $\delta=a-\frac{1}{2}$, we have the result

$$X_I{}^t=-2g\pi\delta\begin{pmatrix}5/2\\0\\1\end{pmatrix}+O(\delta^2) \tag{5.6}$$

at $s=u=t=0$. We therefore have

$$(\epsilon A/B_0)_{\text{model}}=-\delta/b, \tag{5.7}$$

which can be compared to the usual PCAC result

$$(\epsilon A/B_0)_{(\frac{1}{2},\frac{1}{2})\text{ sym. breaking}}=m_\pi{}^2. \tag{5.8}$$

Equating (5.7) and (5.8), we get the result of Lovelace (1968):

$$a+bm_\pi{}^2=\alpha_\rho(m_\pi{}^2)=\tfrac{1}{2}, \tag{5.9}$$

which also guarantees that the model yields the Weinberg scattering length ratio $a_0/a_2=-7/2$ as discussed earlier. In order to make more plausible the identification of (5.4), with $a=\frac{1}{2}$, with the $SU(2)\otimes SU(2)$ symmetric limit, it would be desirable to connect the resonance spectrum and couplings of (5.4) with the mass spectrum and with the matrix elements $\langle H_1 \mid Q_a \mid H_{2\pi}\rangle$ of the axial charge operator resulting from some independent approach. Whether or not this can be done remains an open question.[27]

B. Current Algebra and a Naive Model for πN Scattering

This process is more thoroughly discussed in Sec. X, where we deal with phenomenology. Here we investigate the most naive choice of narrow resonance amplitude for πN scattering with massless pions and ask if it can be made consistent with the restrictions of PCAC and current algebra.

We choose the amplitudes, $A^{(\pm)}(s,t)$ and $B^{(\pm)}(s,t)$, defined by

$$T(\pi_a N\to\pi_b N)=\bar{u}(p')\{(A^++\mathbf{Q}B^+)\delta_{ab}+(A^-+\mathbf{Q}B^-)\tfrac{1}{2}[\tau_a,\tau_b]\}u(p), \tag{5.10}$$

where the momenta are as shown in Fig. 5.1, a and b are isospin indices, $\mathbf{Q}=\frac{1}{2}(q_a+q_b)$, $s=(p+q_a)^2$, and $t=(q_a-q_b)^2$. We choose the invariant amplitudes to be

$$A^{(\pm)}(s,t)=g_1\left(\frac{\Gamma(1-\bar{\alpha}_N(s))\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\bar{\alpha}_N(s)-\alpha_\rho(t))}\pm\frac{\Gamma(1-\bar{\alpha}_N(u))\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\bar{\alpha}_N(u)-\alpha_\rho(t))}\right), \tag{5.11a}$$

$$B^{(\pm)}(s,t)=g_2\left(\frac{\Gamma(-\bar{\alpha}_N(s))\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\bar{\alpha}_N(s)-\alpha_\rho(t))}\mp\frac{\Gamma(-\bar{\alpha}_N(u))\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\bar{\alpha}_N(u)-\alpha_\rho(t))}\right), \tag{5.11b}$$

by analogy with the $\pi\pi$ case. The "trajectory" $\bar{\alpha}_N(x)$ is related to the usual nucleon Regge trajectory by $\bar{\alpha}_N(x)=\alpha_N(x)-\frac{1}{2}$ and we insist our πN amplitude be consistent with (5.4) so that $\alpha_\rho(t)$ is the same rho trajectory which appears in the $\pi\pi$ amplitude above.

The model amplitudes (5.11) are merely a simple first choice. Note that $A^{(\pm)}$ with $\bar{\alpha}_N\to\alpha_\rho$ has the same form as we would use for πK scattering after making an $SU(3)$ rotation on the $\pi\pi$ amplitude (5.4) (Kawarabayashi *et al.*, 1968). The usual asymptotic behavior is included; only the $B^{(\pm)}$ amplitudes contain the nucleon Born terms at $\bar{\alpha}_N=0$, and the nucleon is therefore not parity doubled as it would be if $A^{(\pm)}$ also contained nucleon poles (see MacDowell, 1959). It is important to observe that there are no $I=\frac{3}{2}$ resonance contributions in (5.11) because

$$A_{3/2}{}^s(s,t)=A^{(+)}(s,t)-A^{(-)}(s,t) \tag{5.12}$$

contains no s-channel poles. Therefore, the model contains no analog of the $\Delta(1238)$, $I=\frac{3}{2}$ resonance, which at this stage is "exotic" and must be put in by hand. We will discuss this problem further in Sec. X.

In view of the absence of the $\Delta(1238)$ resonance in the model, we may expect that something will go wrong with the Adler–Weisberger sum rule, which gets most of its contribution from the Δ and associated resonances.[28] It is still interesting to attempt to make (5.11) consistent with $SU(2)\otimes SU(2)$ in the same manner as the $\pi\pi$ model.

Following Dashen and Weinstein (1969b), we find the analog of (5.1) for the transition $H_1\pi_a\to H_2\pi_b$ in the limit $\epsilon\to 0$ (with $m_\pi\to 0$):

$$\tfrac{1}{2}B_0\langle H_2,\pi_a(q_b)\mid S\mid H_1,\pi_a(q_a)\rangle=q_{a\mu}q_{b\nu}\langle H_2\mid\hat{T}(A_a{}^\mu(q_a)A_b{}^\nu(q_b))\mid H_1\rangle+q_{a\mu}\epsilon_{abc}\langle H_2\mid V_c{}^\mu(q_a-q_b)\mid H_1\rangle, \tag{5.13}$$

where the isospin indices are explicit and B_0 is the universal constant of (5.1). The "hat" over the time-ordered product of axial currents indicates that it is formed in the usual way and then has its pion poles removed. From (5.13) we see that the scattering

[27] One possible way to characterize the spectrum, at least in the $\pi-\pi$ case, is to note that it has the l degeneracy of the Schrödinger hydrogen problem, so that it is a realization of an $SO(4,1)$ representation. It is not clear whether this statement is more than descriptive.

[28] See Adler (1956b), Sec. III.

lengths for πN scattering can be computed in the $SU(2)\otimes SU(2)$ symmetric limit in terms of B_0 (Weinberg, 1966). Define the amplitudes

$$F^{(\pm)}(\nu, t)=A^{(\pm)}(\nu, t)+(\nu/2M)B^{(\pm)}(\nu, t), \quad (5.14)$$

where $\nu=\frac{1}{2}(s-u)$ and M is the nucleon mass. For $t=0$ Eq. (5.13) then yields the Adler consistency condition (Adler, 1965a),

$$F^{(+)}(0, 0)=0, \quad (5.15)$$

and the low-energy theorem associated with the Adler–Weisberger sum rule (Adler, 1965b; Weisberger, 1966):

$$(dF^{(-)}/d\nu)(\nu, 0)\mid_{\nu=0}=B_0(1-1/g_A^2). \quad (5.16)$$

We now see if these relations can be satisfied with our simple narrow resonance model (5.11), which yields

$$F^{(+)}(\nu, 0)=(g_1-g_2/2M)$$
$$\times\{[\Gamma(1-\bar{\alpha}_N(s))\Gamma(1-\alpha_\rho(0))/\Gamma(1-\bar{\alpha}_N(s)-\alpha_\rho(0))]+(s\rightarrow u)\}. \quad (5.17)$$

Setting $\nu=0$, (5.15) implies either

$$g_1=g_2/2M \quad (5.18a)$$

or

$$\alpha_\rho(0)=1. \quad (5.18b)$$

Equation (5.18b) conflicts with (5.9) and we are forced to the alternative (5.18a). Equation (5.18a) implies that $\bar{\alpha}_N$ decouples from the system entirely for $t=0$. Clearly this is consistent with (5.16) only if

$$g_A^2=1. \quad (5.19)$$

Recalling that the Adler–Weisberger sum rule relates $(dF^{(-)}/d\nu)(\nu, 0)\mid_{\nu=0}$ to an integral over the difference of the total cross sections $(\sigma_0^+-\sigma_0^-)$ for charged zero-mass pions on protons, we conclude that positivity of widths must be violated somewhere by the model in order to have $g_A^2=1$ since the $I=\frac{3}{2}$ cross section, σ_0^+, is zero.

We can check to see where positivity breaks down. Suppressing isospin, the residues of the poles in the s-channel partial waves of (5.11) are

$$\lim_{\bar{\alpha}_N(s)\rightarrow k}[\bar{\alpha}_N(s)-K]a_{J\pm\frac{1}{2}}{}^J(J, t)$$
$$=\frac{1}{4}\int_{-1}^{+1}dz\left\{(P_{L+1}(z)+P_L(z))R_F(K, t)\right.$$
$$\mp(P_L(z)-P_{L+1}(z))\left[\frac{s_K+M^2}{2Ms_K^{1/2}}\right.$$
$$\left.\left.\times\left(R_A(K, t)+\frac{s_K-M^2}{2s_K^{1/2}}R_B(K, t)\right)\right]\right\}, \quad (5.20)$$

where $\bar{\alpha}_N(s_K)=K$, $L=J-\frac{1}{2}$,

$$R_x(K, t)=x(\nu, t)(\bar{\alpha}_N(s)-K)\mid_{\bar{\alpha}_N(s)=K}, \quad (5.21)$$

and where

$$z=1+[2(K+M^2)/K^2]t. \quad (5.22)$$

[Recall $m_\pi=0$ and note that F in R_F refers to (5.14).] Because of the relation (5.18a), the F term in (5.20) contributes nothing. The remaining portions of the widths are proportional to

$$H_{L\pm\frac{1}{2}}{}^K=\mp\int_{-1}^{+1}dz(1-z)[P_{L+1}'(z)+P_L'(z)]T_K(\tfrac{1}{2}+t), \quad (5.23)$$

where $T_K(x)=\Gamma(x+K)/\Gamma(x)$, $\alpha_\rho(t)=\frac{1}{2}+t$, and we have used the relation [EHF 10.10 (13, 14)]

$$P_L(z)-P_{L+1}(z)$$
$$=(1-z)[P_{L+1}'(z)+P_L'(z)][1/(L+1)]. \quad (5.24)$$

Recalling the behavior of $T_K(x)$ discussed in Sec. III, we see that the forward peak in (5.23) essentially cancels asymptotically. The physical region in t is

$$-K^2/(K+M^2)\leq t\leq 0. \quad (5.25)$$

Between the last zero in $T_K(\frac{1}{2}+t)$ at $t=-K+\frac{1}{2}$ and $t=-K^2/(K+M^2)$, there is a backward peak which is the main contribution to (5.23). The sign of the widths is then

$$\text{sgn}\,(H_{L\pm\frac{1}{2}}{}^K)=\mp(-1)^{K-L}. \quad (5.26)$$

We therefore conclude that all trajectories in the model are parity doubled and one partner of each pair is a ghost, (has negative residue) at least asymptotically. It is the existence of these ghost states which makes the Adler–Weisberger sum rule consistent with $g_A^2=1$ in this model. Clearly we can try to escape from $g_A^2=1$ by adding terms to (5.11) which contain the Δ trajectory. It is possible to remove the first ghost trajectory or even the first two ghost trajectories from the model by adding more terms. This is what is done by those authors whose fits to this amplitude will be discussed in the phenomenology section (Berger and Fox, 1969; Lovelace, 1969b; Fenster and Wali, 1970; Igi and Storrow, 1969), on the grounds that ghost problems occurring on lower trajectories can safely be ignored within the approximation of the model. We do not believe that adding extra terms provides an attractive solution to the ghost-parity doublet problem, because the model does not distinguish leading and nonleading trajectories in any way, and ghosts inevitably remain, at least at the second satellite level.

With respect to this situation Fenster and Wali (1970) make the interesting point that if one works with a narrow resonance amplitude for $\pi N\rightarrow\pi N$ in a circle in s, t, u space with center at the origin and with a radius of a few $(\text{BeV})^2$, and further if one allows enough secondary satellite terms to banish ghosts from such a circle, the resulting g_A^2 has a negative sign.

We will see in Sec. VII that ghost trajectories also occur when we try to construct a self-consistent narrow resonance model for meson–meson scattering, lending credence to the view (Yellin, 1969b) that any attempt to construct a completely self-consistent theory without an infinite number of ghosts will have to go outside the framework of the narrow resonance approximation. The above simplified discussion of πN scattering is very reminiscent of the isospin-factored current algebra theorem of Chang, Dashen, and O'Raifeartaigh (1969), who show that saturating current algebra relations with $I=\frac{1}{2}$ one-particle intermediate states leads to equations which are either physically trivial or deeply diseased.

C. Off-Shell Behavior in the Narrow Resonance Model

In this subsection we comment briefly on the mass extrapolation problem for the reaction $PH \rightarrow P'H'$, where P and P' are pseudoscalar mesons and H and H' are arbitrary hadrons. The point we wish to make is that it is completely *ad hoc* to trivially continue off-shell an on-shell narrow resonance amplitude for $PH \rightarrow P'H'$, containing infinitely rising Regge trajectories, introducing no off-shell dependence on q^2 and q'^2, the $(\text{mass})^2$ of the external pseudoscalar lines. Among the authors who do make this trivial continuation are Arnowitt *et al.* (1966), Cronin and Kang (1969), Frampton (1969), Geffen (1969), Jengo and Remiddi (1969b), Oyanagi (1969), and di Vecchia and Drago (1969). Their results are used in another context by Freund (1969a). It is our opinion that one cannot obtain form factors like $F_\pi(t)$ from the off-shell continuation of a simple Veneziano form.

Specifically, we would like to argue that one cannot trivially compute

$$\langle H \mid J_\mu(x) \mid H' \rangle, \qquad (5.27)$$

the matrix element of an axial or vector current between given hadronic states, by using the on-shell form for $PH \rightarrow P'H'$ directly as a model for the soft meson off-shell dependence of

$$M_{ab}(\nu, t, q_a^2, q_b^2)$$
$$= (q_a^2 - m^2)(q_b^2 - m'^2)\langle H' \mid \hat{T}(D^a(x) D^b(0) \mid H\rangle, \quad (5.28)$$

where $D(x) = \partial^\mu A_\mu(x)$ is the divergence of the axial current and (a, b) are internal symmetry indices. For example, one cannot use a Veneziano form for $PP' \rightarrow PP'$ to calculate the pion electromagnetic form factor $F_\pi(t)$ or the K_{l3} form factors $F_\pm(t)$.

In the next section we will see that if we construct directly a form for (5.27) where one current is linked to a hadron amplitude, then it is necessary to include nontrivial q^2 dependence.

The reason that the trivial off-shell continuation of M_{ab} is precluded is that if there is no q^2 dependence, the Bjorken limit (Bjorken, 1966), which for $P=P'$ and forward scattering is $q_0 \rightarrow \infty$ at fixed $\mathbf{q}$, becomes also the Regge limit $\nu \rightarrow \infty$.

The usual derivation of the Bjorken limit uses the Low equation (Low, 1955)

$$M_{ab}(q_0, \mathbf{q}) = \int [dq_0'/(q_0' - q_0)] \operatorname{Im} M_{ab}(q_0', \mathbf{q}), \quad (5.29)$$

which here reduces to a dispersion relation in ν for fixed t. If one takes $q_0 \rightarrow \infty$ in (5.29), inserts (5.28), and interchanges the q_0 limit with the integral, one gets a series in decreasing integral powers of q_0,

$$M_{ab} = \sum_{n=1}^{\infty} c_n q_0^{-n}, \qquad (5.30)$$

and each of the coefficients is proportional to linear combinations of Fourier transforms of the commutators

$$c_n \propto \text{F.T.}\{[D^a(0), (\partial/\partial x_0)^n D^b(x)]\}. \qquad (5.31)$$

If Regge trajectories are linear, as they are in the Veneziano model, we can always pick t negative enough that the leading trajectory lies lower than (some arbitrary negative integer) $-K$, so that we can compute the first K of the c_n, and the interchange of the $q_0 \rightarrow \infty$ limit and the integral in the Low equation goes through.

The c_n are proportional to the residues of right signature fixed poles (and/or Kronecker-δ singularities) in the J-plane, of the two-current amplitude. Unitarity forces such objects to have zero residue on shell, but in general one expects them to exist off shell where the usual unitarity restrictions do not hold. If we do not introduce a nontrivial q^2 dependence but continue the on-shell narrow resonance model directly, then the c_n will all vanish, and we will have none of the expected right signature fixed poles because the original amplitude does not have them.

This does not quite lead us to a contradiction, but it does tell us that, if we insist on a trivial q^2 continuation, the commutators (5.31) become extremely strange and singular objects, and that the absorptive part of (5.28) is nonzero purely by virtue of the fact that it is given by an infinite sum over the c_n, not a finite sum. In other words, the c_n are analytic functions of t if the theory is local. On the other hand, apart from mathematical complications, we can show by the argument above that they vanish in some neighborhood in t. Therefore they vanish everywhere[29] unless the sum over the c_n diverges in some peculiar fashion or unless some nonlocality is introduced.

These problems do not occur for the classical PCAC and current algebra calculations of the $\pi\pi$ and πN S-wave scattering lengths or the $\pi\pi$ or πN current algebra sum rule, because these involve the knowledge of the chiral symmetry-breaking terms at discrete points only (Dashen and Weinstein, 1969b), while the procedure discussed above would determine them in a

[29] Compare the discussion of wrong signature nonsense-point fixed-pole residues in Sec. III.

neighborhood. If one could compute the detailed behavior of form factors, this would imply a knowledge of the dynamics of chiral symmetry breaking far beyond the usual current algebra–PCAC calculations—and such a step is highly unlikely without a corresponding increase in the depth of our understanding.

The main conclusion we reach from arguments such as those outlined above is that the burden of proof is on the person who proposes an off-shell continuation of the Veneziano model to show that the chosen q^2 dependence is reasonable. Without detailed supplementary assumptions, the narrow resonance model does not itself contain information about off-shell behavior.

D. The Construction of Narrow Resonance, Dual Amplitudes Containing External Vector Currents

In the following section we will discuss the general problem of constructing narrow resonance, dual amplitudes for hadrons coupling to conserved currents. A satisfactory solution to this problem does not presently exist, even in the limited framework we have described in previous sections. However, since the attempt to find such amplitudes necessarily involves many aspects of strong, weak, and electromagnetic interactions, it is highly instructive to sketch out the difficulties one encounters, and we will do that below, using the one- and two-current processes pictured in Fig. 5.2 as specific examples.

Our construction procedure requires the use of the hadronic narrow resonance N-point functions to be discussed in Sec. VII. Here we will merely state the N-point properties we require, referring the reader to Sec. VII for further details.

In attempting to include currents in a narrow resonance world, one hopes to gain some insight into the detailed behavior of weak and electromagnetic form factors, electroproduction structure functions, and the like, or more generally into the behavior of the matrix elements of the weak and electromagnetic currents between arbitrary hadronic states, as a function of momentum transfer. As will become evident below, progress along this line has been ephemeral. In the one-current case no one has found a practical way to constrain q^2 behavior, although one can argue that in principle such a constraint must exist. (In the following the four-momentum of a current will be called q.) In the two-current case a method is at hand, but one is precluded from using it because it has so far proved impossible to make consistent identifications of particles and couplings—i.e., to "factorize"—in all amplitudes properly.

1. One-Current Amplitudes

Following Brower and Weis (1969a, b), we will impose the following conditions on the one-current amplitude $V_\mu(q, p_i)$, shown in Fig. 5.2(a):

(a) $q_\mu V^\mu = 0$;

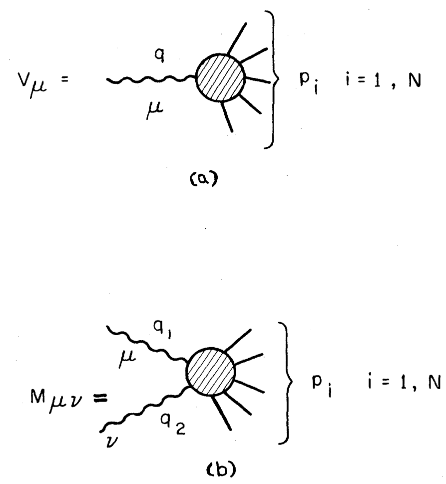

FIG. 5.2(a). The one-current amplitude for N external hadrons. (b). The two-current amplitude for N external hadrons.

(b) Regge behavior in all $s_{ik} = (p_i + p_{i+1} + \cdots p_k)^2$,[30];

(c) meromorphy in q^2 and the s_{ik}, with simple poles for positive real values of the associated invariants;

(d) the residues of poles in s_{ik} are polynomials of finite order in the overlapping variables. (The residues of poles in q^2 are products of a vector meson scattering amplitude times the strength of the current-vector meson coupling.);

(e) the dispersion relations in q^2 and the s_{ik} have no subtractions;

(f) factorization holds, so that the residue of any pole in s_{ik} is a product of some V^μ with a purely hadronic scattering amplitude.

---

[30] The implication here is that one-current amplitudes have no fixed J-plane poles. Discussions of the J-plane properties of one-current amplitudes will be found in Dashen and Lee (1969), Dashen and Frautschi (1966a, b), Mandelstam (1963a, b, c), and Rubinstein, Veneziano, and Virasoro (1968). Dashen and Lee suggested that fixed poles in one-current amplitudes would most easily appear in backward photoproduction of pions. The data of R. L. Anderson *et al.* (1969 and 1968) for backward π^+ photoproduction do not contain evidence for a fixed pole (at $J = \frac{1}{2}$). Neither does the backward π^0 data of D. Tompkins *et al.* (1969). The argument of Dashen and Lee is an illustrative one taken directly from potential scattering, and we will summarize it briefly here.

Consider the T matrix element for

$$\gamma + A \to X + Y, \qquad T \propto \int \psi_{XY}(\mathbf{r}) \exp(i\mathbf{k}\cdot\mathbf{r})\, \boldsymbol{\varepsilon}\cdot\boldsymbol{\nabla}\psi_A(\mathbf{r})\, d^3\mathbf{r},$$

where $\mathbf{k}$ and $\boldsymbol{\varepsilon}$ are the photon momentum and polarization, and ψ_{XY} is the outgoing wave. The statement of Dashen and Lee is that if A is a composite object lying on a Regge trajectory, T will have no fixed poles. They argue as follows: The analytic structure in l of the partial wave amplitudes associated with T is determined by three factors: (i) j_l from the photon's plane wave expansion; (ii) $\psi_{XY,l}$; (iii) ψ_A. The spherical Bessel function j_l is entire in l. The outgoing wave ψ_{XY} can be shown to be equal to the strong interaction S matrix, S_0 times a factor entire in l. The l behavior of ψ_A is not directly relevant. However, if A is elementary, S_0 has a fixed pole, which then appears in ψ_{XY}, and therefore in T. If A becomes composite, in field theoretic language the wave function renormalizer $Z \to 0$, A sits on a Regge trajectory, S_0 has no fixed pole, and neither does T. This leads to condition (b) of the text.

The constraints on analytic behavior in the s_{ik}, contained in (b)–(e), are precisely the same as those already discussed for four-point functions in Secs. II and III, and further extended to N-point functions in Sec. VII. The restrictions on q^2 behavior are the minimal ones required by gauge invariance, analyticity, and factorization.

Given a pure hadronic narrow resonance amplitude for a vector meson coupling to N spinless particles, it is straightforward to write down a function satisfying (b)–(f). We can compute such a function by projecting it from the $(N+1)$-point function for spinless hadrons, using the procedure given in Sec. VII.C (iv). Call this function $B_\mu^{(N,n)}(q, p_i)$, where the label n indicates that the vector meson has mass m_n, and where the q dependence comes from writing the amplitude as a function of the p_i with q determined by energy–momentum conservation. A function satisfying (b)–(f) is then given by

$$V_\mu(q, p_i) = \sum_n g_n(q^2)[m_n^2/(m_n^2-q^2)] \times (g_\mu{}^\nu - q_\mu q^\nu/q^2) B_\nu^{(N,n)}(q, p_i), \quad (5.32)$$

where $\sum g_n(0)=1$, and the $g_n(q^2)$ are as yet undetermined entire functions.[31] We still need to satisfy the gauge invariance condition (a), at $q^2=0$. As shown by Brower and Weis (1969b), Bardakci and Mandelstam (1969), and Fubini and Veneziano (1969), by making a simple restriction of the Regge trajectories, we can get

$$q^\mu B_\mu = 0. \quad (5.33)$$

(The restriction is that the trajectory for a current and K adjacent hadrons be the same as that for the K hadrons alone.) Applying q_μ to (5.32), it is easy to see that (5.33) implies $q_\mu V^\mu = 0$ at $q^2=0$.

It is essential to understand here that all of the pathologies of the pure hadronic N-point functions will appear in the external current amplitudes. In our case this is clear from the structure of (5.32), but it really is a general property. Conversely, because there is only one external current here, for purposes of identification we have really renamed one external line, and if B satisfies factorization, so does our one current amplitude V. As we shall see in Sec. VII, factorization requires at the minimum a degeneracy of hadronic states that increases exponentially as one goes to lower and lower trajectories. This implies, for example, that the nth vector meson in (5.32) is not one particle, but a family of roughly $\exp(n^{1/2})$ individual states, each with a meson-current coupling $g(q^2)$. The thrust of our remarks above is that in the present state of the art there is no practical way of computing the g's, even if they are assumed to be independent of q^2.

The question of q^2 dependence lies at the heart of the difficulties inherent in an ansatz such as (5.32). Explicitly, the viability of the approach we are outlining rests on its ability to yield nontrivial information, for example, about electromagnetic form factors. The problems involved are twofold.

First of all, as mentioned above, one must try to live with all the pathologies of the N-point hadronic bootstrap of Sec. VII. These include a profusion of (ghost) states with imaginary couplings, an utterly unreasonable spectrum, and huge degeneracies of states on the lower-lying Regge trajectories which necessarily follow from the imposition of factorization. Since, as one sums across the vector mesons in (5.32), one reaches lower and lower trajectories, these difficulties are *a priori* very serious, as the large q^2 behavior depends principally on these low-lying levels.

Secondly, even if one could solve satisfactorily the pure hadronic problem, the difficulties inherent in bootstrapping currents, as outlined by Dashen and Frautschi (1966a, b), remain to be confronted. Suppose that we have solved the hadronic problem and that we have in hand the corresponding self-consistent amplitudes. We now want to find all matrix elements $J_{ba}(q^2)$ of a current J, with four-momentum q, between arbitrary hadronic states a and b. Let us assume, as in (e) above, that the scattering amplitudes of the system satisfy unsubtracted dispersion relations in all invariants, including q^2. Consider q^2 to be fixed at some value q_0^2. The *fixed* q^2 dispersion relations then lead to equations of the standard bootstrap form

$$J_{ba}(q_0^2) = \sum_{cd} X_{ba,cd}(q_0^2) J_{dc}(q_0^2),$$

where the matrix X is completely determined by the strong interactions alone. The relations (5.32) determine the $J_{ba}(q_0^2)$ up to an over-all scale factor. Knowing the matrix elements of J for fixed q^2, dispersion relations in q^2 itself can now be used to fix $J_{ba}(q^2)$ for *arbitrary* q^2. However, the arbitrariness in over-all scale for fixed q^2 implies that there is still an over-all unknown entire function in $J_{ba}(q^2)$. If all electromagnetic form factors fall more slowly than exponentially for large q^2, the assumption of unsubtracted dispersion relations in q^2 reduces this over-all unknown entire function to a constant scale factor, which can be fixed by considering amplitudes with more than one external current. On the other hand, if form factors are known or assumed to fall exponentially, (Taylor, 1967; Drell *et al.*, 1967), then we are left with an arbitrary over-all polynomial whose elimination will be extremely difficult.

[31] There is an important remark to be made here closely connected to the discussion of off-shell behavior in Sec. V.C. The point is that, in general, even though there are probably no J-plane fixed poles in the one-current amplitude, analyticity still forces the introduction of nontrivial q^2 dependence. It would be incorrect to substitute, for the B's in (5.32), an on-shell amplitude for a vector meson interacting with N hadrons. The recipe for the q dependence of B given in the discussion preceding (5.32) is absolutely crucial. The problem that arises here is that in the dispersion relation in q^2, if the number of hadrons is greater than 2, there are singularities in q^2 due to the singularities in the s_{ik}. Explicitly, suppose $N=3$, and fix s and t. Then a singularity in u at u_0 gives a contribution to the q^2 dispersion relation at $q^2=u_0+s+t-m_1^2-m_2^2-m_3^2$. This seems to have been first noticed by Mandelstam. See Brower and Weis (1969a), footnote 29.

To summarize the remarks above in terms of the present model, it is possible to write down a dual, narrow resonance, one-current amplitude of which satisfies the required conditions. This is done in (5.32). This ansatz reflects all the ills of the pure hadronic N-point narrow resonance amplitudes and does not yield any nontrivial information about the q^2 behavior of electromagnetic form factors. In principle, according to the remarks above, it ought to be possible to specify completely the model one-current amplitudes, i.e., the g_n in (5.32), given complete knowledge of the pure hadronic amplitudes, provided the resultant form factors decrease more gently than exponentially at large q^2. Since the major result of such a calculation would be to make manifest the model's shortcomings, it is not clear that carrying it out would be particularly useful. Furthermore, the assumptions which are incorporated in (5.32) are general enough that it seems unlikely that mere refinements will solve any of the basic problems. What is necessary is a radical leap in understanding.

2. Two-Current Amplitudes

We now discuss the construction of two-current amplitudes, as pictured in Fig. 5.2(b). Following Brower and Weis (1969b), we first write down a list of requirements, which the reader should compare to the previous list for the one-current case. Let the amplitudes in question be $M_{\mu\nu}^{(\pm)}(q_1, q_2)$. We ask that

(a) $$q_{1\mu}M_{\mu\nu}^{(+)}=0, \tag{5.34a}$$

$$q_{1\mu}M_{\mu\nu}^{(-)}=V_\mu(q_1+q_2), \tag{5.34b}$$

and similarly for q_2. Our currents are assumed to be

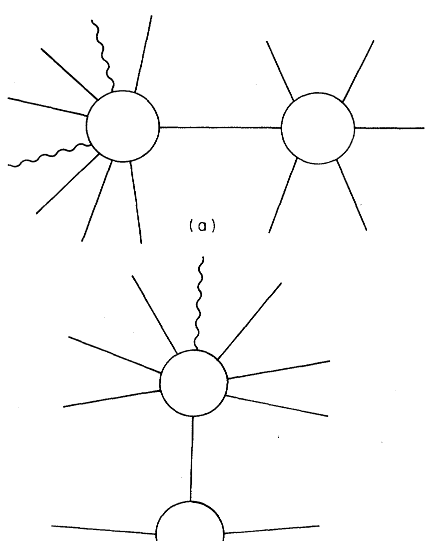

FIG. 5.3. (a) Hadronic factorization. (b) Current factorization.

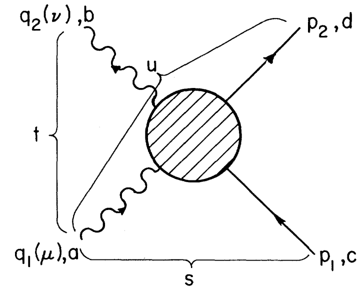

FIG. 5.4. Kinematics for virtual Compton scattering.

isovectors so that we can form the even and odd isospin combinations

$$M_{\mu\nu}^{(\pm)}=\tfrac{1}{2}[M_{\mu\nu}^{ba}\pm M_{\mu\nu}^{ab}]; \tag{5.35}$$

(b) Regge behavior in all variables except possibly those overlapping the two-current channel;

(c) same as assumption (c) for the one-current amplitude;

(d) the residue of a pole in q_1^2 is a one-current amplitude for the production of a vector meson;

(e) same as (e) for the one-current amplitude;

(f) we have two types of factorization, hadronic factorization and current factorization, as shown in Fig. 5.3(a) and (b).

There are important differences between the list above and that for the one-current case. First of all, in item (a) we have replaced the simple gauge invariance requirement with the full current algebra divergence conditions. Equation (5.34b) tells us that taking the divergence of the part of the amplitude with isospin 1 in the two-current channel returns us to the one-current case which we considered above, without affecting any of the hadrons. Together, the relations (5.34) imply that the commutation relations (Gell-Mann, 1964a)

$$\delta(x_0)[V_0^a(x), V_\nu^b(0)]=i\epsilon_{abc}V_\nu^c(x)\delta^4(x) \tag{5.36}$$

hold for the current densities. There is no constraint on the form of the space–space commutation relations implicit in the list of requirements.

In requirements (b) and (d) we now see the echo of the discussion in Sec. V.C. Once we construct our M, we have in hand an off-shell continuation of the amplitude for vector meson+vector meson→N spinless hadrons. For definiteness, let $N=2$. Then we have the off-shell continuation (see Fig. 5.4):

$$i\int d^4x\langle p_2\,|\,\tilde{T}(V_\mu^a(\tfrac{1}{2}x), V_\nu^b(-\tfrac{1}{2}x))\,|\,p_1\rangle e^{-iQx}=M_{\mu\nu}^{ba}(P, Q, q), \tag{5.37}$$

where

$$Q=\tfrac{1}{2}(q_1+q_2);\qquad q=\tfrac{1}{2}(q_1-q_2);\qquad P=\tfrac{1}{2}(p_1+p_2) \tag{5.38}$$

and where $\tilde{T}$ is the covariant form of the time-ordered product. It is well known that the Compton amplitude in (5.37) in general has t-channel fixed poles in J (Bronzan *et al.*, 1967; Singh, 1967) and also J-plane Kronecker delta singularities (Doesch and Gordon, 1968; Gross and Pagels, 1968). The two-current amplitude contains more than moving poles in the J plane, just as implied by the arguments in Sec. V.C. An explicit construction which makes this point clear has been performed by Brower, Rabl, and Weis (1970), for virtual Compton scattering of currents off pions.

From (5.37) it is straightforward to check that (5.34) implies (5.36). Given an explicit choice of amplitude, one can also compute the space–space commutators,[32] and the behavior of $M_{\mu\nu}$ as Q^2 becomes large, which is relevant for inelastic scattering and electromagnetic mass differences. With regard to this last subject we will confine ourselves to the observation that any naive choice of amplitude satisfying (a)–(f) will have a Regge-behaved large Q^2 limit, of the form $(Q^2)^{\alpha(\nu)}$, where $\nu\equiv\frac{1}{2}(s-u)$. One would like, of course, to get instead the scaling limit of Bjorken (1969), which says that as Q^2 becomes large and negative (very spacelike in our metric), the electroproduction structure functions become dependent on the ratio ν/Q^2 alone.

Last and most crucial in the list of requirements is the factorization condition (f). There are two types of factorization, as shown in Fig. 5.3, hadronic factorization and current factorization. If this last kind of factorization did not exist, then a solution of the hadronic factorization problem for the scattering of two vector mesons to make N hadrons would automatically ensure that the two-current amplitude also factorized, just as in the one-current case. Current factorization will now restrict the form factors provided one can enforce it for nonleading trajectories. To our knowledge this has *not* yet been done, and there is no existing model which actually satisfies (a)–(f) for two currents. Of course, just as in the one-current case, all pathologies of the narrow resonance N-point amplitudes appear here.

For various explicit two-current models the reader may refer to the work of Ademollo and del Giudice (1969), Bander (1969), Brower, Rabl, and Weis (1970), Ohba (1969), and Sugawara (1969). None of these cures the basic pathologies discussed above[33] nor is it likely that anyone will find a satisfactory model that is not *ad hoc* for deep inelastic electron scattering, since this is closely tied to factorization of nonleading trajectories.

Additional material relevant to this section can be found in Abers and Teplitz (1969), Ahmad, Fayyazuddin, and Riazuddin (1969), de Alwis *et al.* (1969), Amati, Jengo, Rubinstein, Veneziano, and Virasoro (1968), Brandt (1969), Brower and Halpern (1969), Cooper (1970), Costa (1969), Drago (1969), Fujisaki (1969a, b), Freund and Rivers (1969), Goldberg and Srivastava (1969), Hsu (1969), McKay and Walter (1969), Osborne (1969), Savoy (1969), Schnitzer (1969), and Zee (1969).

VI. ALTERING THE NARROW RESONANCE APPROXIMATION

As pointed out in Sec. II, the single most unphysical characteristic of the narrow resonance model is the presence of poles on the real axis of the Mandelstam invariants and the absence of physical normal threshold cuts. We have discussed the interpretation of the narrow resonance limit in terms of FESR's and we now want to examine methods of extrapolating away from the narrow resonance limit to obtain the properties of physical amplitudes. In making this extrapolation we would like to preserve as many of the desirable properties of the Veneziano model as possible. For example, we would want the finished product to have Regge asymptotic behavior and crossing. Certain other properties of the model cannot hold exactly when we have physical amplitudes, and we are interested in how they are altered.

If we do not have narrow resonances, the Regge trajectories can no longer be exactly linear, but we would like to maintain a situation where the real part of $\alpha(s)$ is approximately linear in those regions where particles have been found empirically to lie on straight lines in Chew–Frautschi plots. We therefore assume that the trajectory functions satisfy a once-subtracted dispersion relation (Cheng and Sharp, 1963),

$$\alpha(s)=a+bs+\pi^{-1}(s_0-s)\int_{s_0}^{\infty}\frac{\operatorname{Im}[\alpha(s')]}{(s-s')(s_0-s')}\,ds', \quad (6.1)$$

and that in the low-energy region the contribution of the integral is small.

In view of the role of dispersion relations in the derivation of FESR's, we would also like to require that there exist a region where the amplitude is completely determined by an unsubtracted fixed variable dispersion relation:

$$A(s,t)=\pi^{-1}\int_{-\infty}^{u_0}\frac{\operatorname{Im}A(s,t)}{u-u'}\,du' +\pi^{-1}\int_{s_0}^{\infty}\frac{\operatorname{Im}A(s,t)}{s-s'}\,ds'. \quad (6.2)$$

[32] The procedure for calculating the space–space commutators given an explicit two-current amplitude is as follows: Take the Bjorken limit (Bjorken, 1966) $|Q_0|\to\infty$, P_μ, q_μ fixed of M in Eq. (5.37):

$$M_{\mu\nu}{}^{abcd}\underset{Q_0\to\infty}{\longrightarrow}\int d^3\mathbf{x}\exp(i\mathbf{Q}\cdot\mathbf{x})\cdot\sum_{n=0}^{\infty}\frac{(-i)^n}{Q_0{}^{n+1}} \cdot\langle p_2,d\,|\left(\frac{\partial}{\partial x_0}\right)^n V_\mu{}^a(0,\tfrac{1}{2}\mathbf{x}),\qquad V^b(0,-\tfrac{1}{2}\mathbf{x})\,|\,p_1,c\rangle + \text{polynomial in } Q_0.$$

The quantity we want is the space–space piece of the coefficient of $1/Q_0$.

[33] Some of the alternative models mentioned above have "good" large Q^2 behavior, in return for which they acquire other undesirable features. See Brower, Rabl, and Weis (1970) for a discussion.

This assumes a combination of analyticity and a power bound and is the generalization of the concept of atonous duality discussed in Sec. II. If the amplitude satisfies an unsubtracted dispersion relation, questions concerning resonance dominance of the discontinuity, the conjectures of Freund (1968b) and Harari (1968), and the neglect of Regge cuts in the FESR's can be discussed after we have constructed a model for the continuation away from the narrow resonance limit.

A. Complex Trajectories and Ancestors

Attempts to insert complex trajectories directly into the Veneziano model (Roskies, 1968; Paciello *et al.*, 1969b) generate finite total widths for the resonances but also result in an infinite tower of spins at each resonance mass. This is because the residue of a pole in s, in the model, is a polynomial in $\alpha(t)$ rather than being a polynomial in t itself. In general, the integral of $P_L(z)$ times some complicated function, not a polynomial, is nonzero for all L. Resonances with spin greater than $\mathrm{Re}\,\alpha(t)$ are called ancestors.

Another problem is that this simple procedure gives all poles at a given mass the same total width regardless of the elastic width predicted by angular momentum projections. For example, if we give a phenomenological width to the ρ-trajectory in the simple $\pi\pi$ amplitude discussed in Sec. III, then the ϵ resonance will have the same total width as the ρ in contrast to the partial width ratio:

$$\Gamma_{\text{elastic}}(\epsilon)/\Gamma_{\text{elastic}}(\rho)=\tfrac{9}{2}. \tag{6.3}$$

This degeneracy of total widths makes it impossible to calculate meaningful phase shifts from this simple approach. On the other hand, the ancestor problem is not necessarily fatal since the coupling to these high spin states is usually small.

B. K-Matrix and Crossing

Clearly, if we intend to take seriously the elastic widths for resonances predicted by the Veneziano model, we must have a method of displacing the poles from the real axis which depends on the angular momentum structure. A simple way of doing this is the K-matrix method suggested by Lovelace (1969a). Lovelace suggests that we interpret the partial wave projections of the Veneziano model, $a^I(J,t)$, as the K-matrix elements of the physical partial wave projections, $f^I(J,t)$:

$$f^I(J,t)=a^I(J,t)/[1+\rho(t)a^I(J,t)]. \tag{6.4}$$

Elastic unitarity gives the imaginary part of $\rho(t)$, in a channel with mass m_1 and m_2, as

$$\mathrm{Im}\,\rho(t)=-\left[\frac{t-(m_1-m_2)^2}{t-(m_1+m_2)^2}\right]^{-1/2}\Theta[t-(m_1+m_2)^2]$$
$$-\left[\frac{t-(m_1-m_2)^2}{t-(m_1+m_2)^2}\right]^{1/2}\Theta[(m_1-m_2)^2-t], \tag{6.5}$$

and the real part is chosen by assuming $\rho(t)$ satisfies an unsubtracted dispersion relation so that

$$\mathrm{Re}\,\rho(t)=\frac{(m_1^2-m_2^2)\ln(m_1/m_2)}{\pi t}$$
$$-\frac{2(m_1+m_2)^2}{\pi t}\left[\frac{t-(m_1-m_2)^2}{t-(m_1+m_2)^2}\right]^{1/2}$$
$$\times\ln\left\{\left(\frac{t-(m_1+m_2)^2}{4m_1m_2}\right)^{1/2}+\left(\frac{t-(m_1-m_2)^2}{4m_1m_2}\right)^{1/2}\right\}. \tag{6.6}$$

This method essentially gives all resonances in the model a total width equal to their elastic width and can therefore be presumed to be approximately correct, if at all, below the first inelastic threshold.

This may be an improvement over simply inserting complex trajectories in the beta function but the predictions cannot be completely reliable, even below the first elastic threshold, if crossing symmetry plays an important role. If the original narrow resonance amplitude has crossing symmetry, then the K-matrix form destroys this property, as can be seen by recalling that amplitudes satisfying exact elastic unitarity, and therefore containing no production processes, cannot simultaneously satisfy analyticity and crossing (Aks, 1965).

In fact, crossing symmetry plays an important role in determining the low-energy resonance parameters of the $\pi\pi$ system, so that the low-energy K-matrix phase shifts predicted by Lovelace (1969a) cannot be completely consistent. We will return to this question in Sec. X where we discuss phenomenology.

Arbab (1969) has also proposed a unitarization scheme based on the form of the Veneziano partial wave amplitude. He finds unitary threshold corrections to the reduced residue function of the leading Regge pole. These corrections destroy the crossing properties of the model so this method has the same drawback as the K-matrix scheme and is subject to the same criticism although the details of the scheme are different.

Balazs (1969) and Atkinson *et al.* (1969) have taken an approach where the lowest pole in the Veneziano model is replaced by a finite cut with a unitary discontinuity. This is used as input in an N/D calculation, where the far-off singularities are given by the unmodified Veneziano form. The method is more complicated than the K-matrix approach, but it is not clear that it is an improvement. More work needs to be done if this approach is to be evaluated, with the emphasis on including coupled channels and maintaining crossing symmetry.

C. Smearing

All the attempts discussed above to unitarize the Veneziano model are based on rather traditional methods of calculation and emphasize the low-energy, elastic unitarity region. Of more immediate interest are models invented by Martin (1969) and by Suzuki

(1969) for extending the Veneziano model away from the narrow resonance limit while maintaining crossing symmetry and polynomial residues. These methods are not, in themselves, unitarization schemes but are ways of removing the outstanding single nonunitary property of the model, the zero-width resonance.

Martin treats the δ-function discontinuities present in the Veneziano model with a standard convolution procedure familiar in distribution theory. He takes the Veneziano amplitude for $\pi\pi$ scattering, Eq. (3.13), and smears its trajectory slope with a test function, $\phi(b)$, which is positive and vanishes at the end points of the integration:

$$\tilde{F}_0(s,t)=\int_{x_0}^{1} dx\phi(x)\,\frac{\Gamma(1-a-bxs)\Gamma(1-a-bxt)}{\Gamma[1-2a-b(s+t)x]}\,. \tag{6.7}$$

For a suitable $\phi(x)$, the poles in Eq. (6.7) are displaced from the real axis onto the second sheet. Martin's amplitude does not have purely power behavior. Instead, the asymptotic behavior is modified by a logarithmic factor indicating the presence of a cut in the J plane. Since it is almost certain that a unitary amplitude contains Regge cuts, this type of behavior is certainly not undesirable although, for aesthetic reasons, it would probably be preferable if the leading singularity in each channel remained a pole, and cuts only appeared in nonleading order.

The location of the resonance poles in Eq. (6.7) is given by the effective Regge trajectory,

$$\bar{\alpha}(s)=a+b(x_0+i\Gamma)s, \tag{6.8}$$

where x_0 and Γ are determined by the form of $\phi(x)$. This effective trajectory is not real below threshold in s, but the amplitude (6.7) has the correct threshold behavior so that this is not necessarily an objection to the smeared form.

Bali, Coon, and Dash (1969a) and Huang (1969) have developed slightly different smearing schemes. All these approaches share the flaw that the total widths continue to be the same for all resonances of a given mass.

Suzuki's approach to generating finite width resonances is most conveniently expressed in terms of the integral representation of the beta function. If we introduce the complex trajectory function [see Eq. (6.1)]

$$\alpha(s)=a+bs+\Delta(s) \tag{6.9}$$

into the amplitude in the form

$$\tilde{B}(s,t)=\int_0^1 dz z^{-\alpha(s)-1+\Delta(s)f(z)}(1-z)^{-\alpha(t)-1+\Delta(t)f(1-z)}, \tag{6.10}$$

where $f(0)=0$ and $f(1)=1$, we find that, in order to insure Regge behavior for $|s|\to\infty$, arg $s\epsilon(\delta, 2\pi-\delta)$, and to guarantee the absence of ancestors, we must restrict

$$\lim_{|s|\to\infty,\, on\ 1st\ sheet} \Delta(s)/s=0 \tag{6.11}$$

and require that all derivatives of $f(z)$ vanish at $z=0$, 1[34]:

$$d^n f(z)/dz^n\,|_{z=0,1}=0. \tag{6.12}$$

While Suzuki's model tackles the large unitary violations caused by the physical sheet poles, in contrast to Martin's approach it does not introduce J-plane cuts but maintains pure pole behavior. Again, it has the flaw that total widths are degenerate. There is an infinite class of functions which satisfy the constraints (6.12) and each of them generates a slightly different relation between the pole parameters and the discontinuity across the cut in the amplitude.

VII. THE NARROW RESONANCE BOOTSTRAP

We would like to review attempts to construct a self-consistent set of narrow-resonance, hadronic amplitudes (Mandelstam, 1969b, 1970a, b; Bardakci and Halpern, 1969). The program for this bootstrap scheme is as follows:

(a) One constructs an infinite set of atonous dual, crossing symmetric, narrow resonance, Regge-behaved amplitudes, for arbitrary numbers of external particles.

(b) One imposes self-consistency on the system, in the form of factorization of pole residues.

(c) The amplitudes constructed as in (a) and (b) are treated as Born terms of a complete theory.[35] One attempts to escape from the unitarity violations of the narrow resonance model by forming a "Reggeized perturbation expansion" of diagrams containing closed loops (Kikkawa, Sakita, and Virasoro, 1969).

In this section, we will discuss steps (a) and (b),

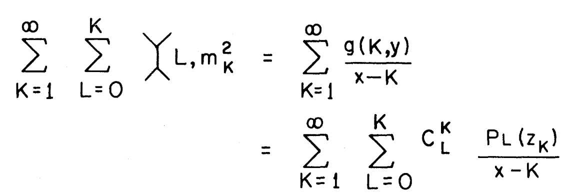

FIG. 7.1. Narrow resonance four-point function for $\pi\pi\to\pi\pi$ as sum over Feynman tree graphs with definite internal angular momentum.

[34] This function can be thought of as the limit

$$f(z)=\lim_{M\to\infty}[B_z(M,M)/B(M,M)],$$

where B_z is the incomplete beta function. See Suzuki (1969) for a more thorough discussion of the derivation.

[35] There is a difference among the practitioners as to whether one takes the N-point functions to be Born terms in the general or in the specific field theoretic sense. For the doubting group, which includes the authors, who look upon the poles of the narrow resonance amplitudes as bound states, neither meaning makes sense.

limiting ourselves to a system of mesons only. Mandelstam (1970a, b) has extended the discussion to baryons, and we will comment briefly in Sec. VIII on some peripheral matters associated with that problem. We will return to the details involved in step (c) in Sec. IX.

In order to clarify what is involved in implementing (a) and (b), we would like to pose the following unsolved problem, which we will refer to as the "inside–outside" problem. As emphasized in Sec. III, the general structure of the narrow resonance model for $\pi\pi\rightarrow\pi\pi$ is such that the amplitude can be represented by an infinite sum over Feynman tree diagrams in a given channel as in Fig. 7.1 and Eq. (3.18).

The amplitude contains internal states labeled according to (mass)^2, K, and angular momentum, L, whose coupling to the $\pi\pi$ system is given by $[c_L{}^K]^{1/2}$, where $c_L{}^K$ is defined by

$$F(x,y)=\sum_{K=1}^{\infty}\frac{g(K,y)}{x-K}=\sum_{K=1}^{\infty}\sum_{L=0}^{K}\frac{c_L{}^K P_L(\cos\theta_x)}{x-K}\ . \quad (7.1)$$

The inside–outside problem is posed as follows: Treat all internal states in (7.1) as external states, and form all possible N-point functions consistent with the original $\pi\pi$ amplitude. For example, the $\pi\pi\rightarrow\pi\pi$ amplitude contains an internal ρ state ($K=1$, $L=1$). We can try to form amplitudes for $\rho\pi\rightarrow\rho\pi$ and $\rho\rho\rightarrow\rho\rho$ consistent with $\pi\pi\rightarrow\pi\pi$, and so on for other internal resonances until the system closes and all states appear externally and internally for 4 external lines, 5 external lines, etc. If we could solve this problem, we would have a set of N-point functions consistent with (a) and (b) above, containing all narrow resonance poles symmetrically as tree poles and as external scattering states.

The inside–outside problem has not been solved, but preliminary work seems to indicate that the particle spectrum will not be free of negative residue states (ghosts) so that we will have to abandon the positivity condition used in Sec. III in order to find any solution at all. The explicit ansatz of Mandelstam (1969b) and Bardakci and Halpern (1969), which we will discuss below, satisfies conditions (a) and (b), but it neither solves completely an inside–outside problem nor excludes ghosts.

The Mandelstam–Bardakci–Halpern ansatz is formulated in terms of the quark model. It has the following pathologies:

(1) All trajectories are parity doubled, one partner being a ghost.

(2) There is an infinity of trajectories with abnormal C.

(3) The π and ρ mesons are degenerate and the system does not choose the Goldstone $SU(2)\otimes SU(2)$ realization.

(4) Factorization leads to an exponential degeneracy of the lower lying trajectories.

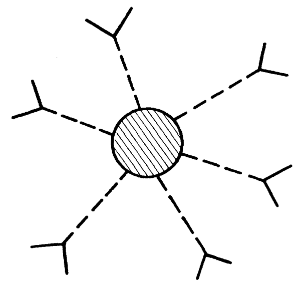

FIG. 7.2. Quark content of the narrow resonance ansatz of Bardakci and Halpern (1969). The solid, external lines are quarks, and the dotted lines are quark–antiquark bound states (mesons) which form the internal states of the system.

(5) The model requires unobserved trajectories.

(6) The structure of the resultant amplitudes leads to an indefinite metric, and to two different infinite families of ghosts.

Diseases (1), (2), (4), and (5) are probably intrinsic to the narrow resonance bootstrap. Disease (3), which ruins soft pion applications, may be a particular property of this ansatz only. It would be interesting to see if the inside–outside problem allows a solution with $m_\delta > m_\pi$ and $m_\pi=0$. Disease (6) arises, as we shall see below, from the half-integral spin of the quarks used as meson building blocks, plus the rather *ad hoc* construction procedure, and also from the forcing of factorization.

The actual construction procedure is to separately solve the problems of internal symmetry, ordinary spin, and orbital angular momentum, and then present the final result as a product of three factors. This procedure enables one to carry through (a) and (b) in a manner which is interesting and illustrative, but probably unphysical. The separation of the spin and orbital factors, in particular, does not occur in any known set of Feynman diagrams, say in quantum electrodynamics, and results in an indefinite metric, as in (6) above.

Specifically, we consider a $2N$-point function, where the external lines represent N quarks and N antiquarks and write the total amplitude as

$$A_{2N}=T_{2N}S_{2N}B_{2N}, \quad (7.2)$$

where T, S, and B are the isospin, ordinary spin, and orbital factors, respectively.[36] The resulting mass spectrum can then be classified by $SU(6,6)\otimes O(3)$ and the vertices by $SU(6)_W$.[37]

We will consider here only the meson bootstrap, in

[36] The reader may find it instructive to consider in this connection the result of trying to bootstrap the bound states of positronium in this manner, keeping only Feynman diagrams with multiphoton exchange, just as in the eikonal approximation of Abarbanel and Itzykson (1969), Chang and Ma (1969), and Levy and Sucher (1969).

[37] In other words we have a spectrum-generating algebra. See Dothan, Gell-Mann, and Ne'eman (1965).

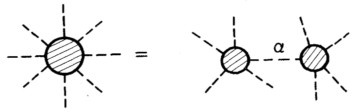

FIG. 7.3. Factorization of $SU(3)$ factor T_{2N} in (7.2). The dotted lines represent the internal meson states in the narrow resonance ansatz.

which the amplitude A_{2N} can be thought of in the form of Fig. 7.2, in which each quark–antiquark pair forms into a meson, and at the end the quarks are thrown away leaving the amplitude we want. The extension to baryons has been considered by Mandelstam (1970a); for a discussion we refer the reader to his paper and to that of Olesen (1970a).

A. The $SU(3)$ Problem

The original solution of this problem was given by Chan and Paton (1969). We will follow here the arguments of Bardakci and Halpern (1969). Each external line in Fig. 7.2 is to be associated with a quark wave function which factorizes as above into internal symmetry, spin, and orbital pieces. Choose $SU(3)$ as the internal symmetry and call that part of the quark wave function $\chi(i)$, where i labels the quarks. We will consider mesons only and will force the particles to transform as $\mathbf{1}\oplus\mathbf{8}$ under $SU(3)$ with $\mathbf{3}$ and $\mathbf{3}^*$ being the quark and antiquark representations. If we form the quark–antiquark 3×3 matrix $\chi(i)\chi^\dagger(i)$, we can write

$$\chi(i)\chi^\dagger(i)=\sum_{\alpha=0}^{8}[\chi(i)\lambda_\alpha\chi^\dagger(i)]\lambda_\alpha \tag{7.3}$$

in terms of the nine λ_α's of Gell–Mann (1961).

Now, defining the coefficients as

$$(C_\alpha)_i=\chi^\dagger(2i)(\lambda_\alpha)_i\chi(2i-1), \tag{7.4}$$

where there is no sum on i, the isospin factor is

$$T_{2N}=\mathrm{Tr}\,\{\prod_{i=1}^{N}(C_\alpha)_i(\lambda_\alpha)_i\}. \tag{7.5}$$

The choice of indices in (7.4) tells us that (7.5) is cyclically symmetric in the hadron labels i. At a meson pole $(C_\alpha)_i=1$, and all this boils down to

$$T_{2N}=\mathrm{Tr}\,[\prod_{i=1}^{N}\lambda_{\alpha_i}], \tag{7.6}$$

where the ith *meson* has $SU(3)$ index α_i.

The expression (7.6) factorizes properly because of the identity between arbitrary $SU(3)$ matrices,

$$\mathrm{Tr}\,(AB)=\sum_{\alpha=0}^{8}\mathrm{Tr}\,(A\lambda_\alpha)\,\mathrm{Tr}\,(\lambda_\alpha B), \tag{7.7}$$

which just tells us that the $SU(3)$ amplitude can be written as in Fig. 7.3. Now we pass to the spin problem.

B. Spin Structure

We take for the ordinary spin wave function of the quark, the Dirac spinor $u(i)$. By analogy with the above we define

$$(S_p)_i=\bar u(2i)(\Gamma_p)u(2i-1), \tag{7.8}$$

where the Γ_p are the 16 Dirac matrices, and the spin factor becomes

$$S_{2N}=\mathrm{Tr}\,\{\prod_{i=1}^{N}(S_p)_i(\Gamma_p)_i\}. \tag{7.9}$$

In (7.9) the trace is over the Dirac space. The meson "propagator" is then

$$S_4=S_1S_2+P_1P_2+\bar P_1\bar P_2+(V_\mu)_1(V_\mu)_2-(\bar V_\mu)_1(\bar V_\mu)_2 -(U_\mu)_1(U_\mu)_2-(\bar A_\mu)_1(\bar A_\mu)_2, \tag{7.10}$$

which contains the following set of trajectories:

$0^{++}(S)$, $0^{-+}(P)$, $1^{--}(V)$, $1^{--}(\bar V)$, $1^{+-}(\bar U)$, and $1^{+-}(\bar A)$,

where J^{PC} labels the quantum numbers of the lowest particle on the trajectory. Let $(q_\nu)_i=q_\nu(2i)-q_\nu(2i-1)$ and let m be the quark mass. Then we have

$$(q_\nu)_i(\bar V_\nu)_i=0=(q_\nu)_i(U_\nu)_i, \tag{7.11}$$

$$(q_\lambda)_i(A_\lambda)_i=0. \tag{7.12}$$

We use the usual names $\{\Gamma\}=\{S, P, T, A, V\}$, and the associated quantities $\bar A$, $\bar U$, $\bar V$, $\bar P$ are defined by

$$(A_\mu)_i=u(2i)\gamma_5\gamma_\mu u(2i-1)=(\bar U_\mu)_i+[2m(q_\mu)_i/q^2]P_i, \tag{7.13}$$

$$(T_{\mu\nu})_i=[1/(q_i^2)^{1/2}]\{(q_\mu)_i(V_\nu)_i-(q_\nu)_i(V_\mu)_i +i\epsilon_{\mu\nu\lambda\rho}(q_\lambda)_i(\bar A_\rho)_i\}, \tag{7.14}$$

$$(\bar V_\nu)_i=[(q_\mu)_i/(q_i^2)^{1/2}](T_{\mu\nu})_i, \tag{7.15}$$

and

$$\bar P_i=[1/(q_i^2)^{1/2}]u(2i)\gamma_5\mathbf{q}_iu(2i-1)=[2m/(q_i^2)^{1/2}]P_i. \tag{7.16}$$

To complete the argument we need the identity in

TABLE 7.1 Leading trajectories in narrow resonance dual quark model.

J^P	C	Ghosts?
0^+	+	Yes
0^+	−	Yes
0^-	+	No
0^-	+	No
1^-	−	No
1^-	−	No
1^+	−	Yes
1^+	−	Yes

Dirac space,

$$\text{Tr}\,(AB)=\sum_p \text{Tr}\,(A\Gamma_p)\;\text{Tr}\,(\Gamma_p B) \qquad (7.17)$$

projecting out definite spins as above. This object is found to contain the trajectories listed, plus two more which do not explicitly appear in the 4-quark amplitude: (a) an extra pion trajectory coupling as $\gamma_5\boldsymbol{q}/q$; (b) an abnormal scalar trajectory coupling as $\boldsymbol{q}/q$, arising from the divergence of the γ_μ term.

Explicitly writing out (7.17), with $\gamma_{\mu A}\equiv\gamma_\mu-[q_\mu(A)\boldsymbol{q}(A)/q^2(A)]$, we have

$$\begin{aligned}\text{Tr}\,(AB)=&\ \text{Tr}\,(A)\ \text{Tr}\,(B)+\text{Tr}\,(A\gamma_5)\ \text{Tr}\,(\gamma_5 B)\\&+\text{Tr}\,(A\gamma_{\mu A})\ \text{Tr}\,(\gamma^{\mu B}B)\\&-\text{Tr}\,(A\cdot\boldsymbol{q}(A)/q(A))\ \text{Tr}\,(\boldsymbol{q}(B)/q(B)\cdot B)\\&-\text{Tr}\left(A\,\frac{\gamma_5\gamma_{\mu A}}{q(A)}\right)\text{Tr}\left(\frac{\gamma_5\gamma^{\mu B}}{q(B)}\,B\right)\\&+\text{Tr}\left(\frac{A\gamma_5\boldsymbol{q}(A)}{q(A)}\right)\text{Tr}\left(\frac{\gamma_5\boldsymbol{q}(B)B}{q(B)}\right)\\&-\text{Tr}\left(\frac{A\sigma_{\mu\nu}q^\nu(A)}{q(A)}\right)\text{Tr}\left(\frac{\sigma^{\mu\lambda}q_\lambda(B)B}{q(B)}\right)\\&-\text{Tr}\left(\frac{A\gamma_5\sigma_{\mu\nu}q^\nu(A)}{q(A)}\right)\text{Tr}\left(\frac{\gamma_5\sigma^{\mu\lambda}q_\lambda(B)B}{q(B)}\right),\end{aligned} \qquad (7.18)$$

where we imagine we have a scattering process with two blobs, A and B, connected by an internal line carrying momentum $q_\mu(A)=-q_\mu(B)$, with $q(A)=q(B)=[q^2(A)]^{1/2}=[q^2(B)]^{1/2}$ being the mass of the object exchanged between A and B. The decomposition (7.18) explicitly resolves the amplitude into pure J^{PC} pieces, where $J=0$ or 1.

Now, to finish off this discussion, we want to identify the ghosts. The proper Feynman vertices (Bjorken and Drell†, 1969) for the particles under consideration are, for S, P, V, and A,

$$\{-i,\ \boldsymbol{q}/q\},\ \{\gamma_5,\ \gamma_5\boldsymbol{q}/q\},\ \{-i(\gamma_\mu-q_\mu\boldsymbol{q}/q^2),\ \sigma_{\mu\nu}q^\nu/q\},$$
$$\text{and}\quad \{-i(\gamma_5\gamma_\mu-q_\mu\gamma_5\boldsymbol{q}/q^2),\ -i\gamma_5\sigma_{\mu\nu}q^\nu/q\}, \qquad (7.19)$$

respectively. If we square the phase factors and compare with (7.18), we have the final list of trajectories shown in Table 7.1. As can be seen there, an infinite family of ghost trajectories has appeared.

There are two π and two ρ trajectories in this model. In principle, they are identical pairs. If we introduce two mixing angles, Θ_π and Θ_ρ, into the system, there is a choice of Θ_π and Θ_ρ, used by Mandelstam (1969b), which turns out to force precisely $SU(6)_W$ symmetry for the meson–meson–meson vertices.

C. The Orbital Factor

To complete the discussion we need to find an amplitude which will serve as the orbital factor in (7.2). This clearly is the same problem as the construction of a $2N$-point function for scalar particles since the spin has been factored out as above. We will now discuss the problem of directly generalizing Veneziano's four-particle form (Veneziano, 1968) to M particles. The solution of this problem yields an amplitude, B_M, which we will use in (7.2) for $M=2N$.

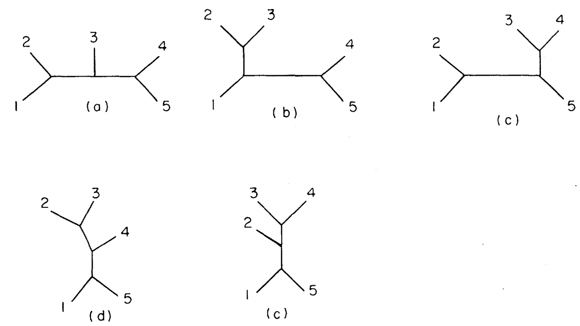

FIG. 7.4. The five different planar Feynman tree diagrams possible for a given ordering, 12345, of five particles.

The structure of the orbital problem is more complex than the spin and internal symmetry problems because many nontrivial constraints are involved. The first step in its solution occurred when Bardakci and Ruegg (1968) and Virasoro (1969a) generalized Veneziano's model to five particles. Chan (1969), Goebel and Sakita (1969), and Koba and Nielsen (1969) then extended this form to the case of N external particles.

To discuss this generalization we restrict attention to an idealized system of neutral bosons which can later be interpreted as spinless quarks for the purpose of the bootstrap. This system is defined by one parent Regge trajectory, the lowest member of which is a massive particle of spin-parity $J^P=0^+$. The parent trajectory is therefore restricted to have a negative intercept. We will first explain the concepts of planar diagrams, overlapping channels, and the necessity for particle ordering. Next, we will construct B_N, an explicit, nonunique, example of an N-particle narrow resonance amplitude. We will examine the multi-Regge limits of B_N and its factorization properties and, finally, discuss its use in the bootstrap scheme mentioned above.

(i) *Planar Diagrams, Overlapping Channels, and Tree Diagrams*

To define the model for the scattering of N of these spinless particles we are going to construct functions with the singularity structure of planar Feynman tree diagrams. That is, planar diagrams with three particle vertices and without internal loops. Figure 7.4 shows the different Feynman tree diagrams which will be present in the model for a given ordering of five particles. The ordering of the particles is crucial since for a given ordering *planar* Feynman tree diagrams can be constructed with poles in each of the planar Mandelstam

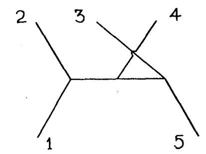

FIG. 7.5. Example of a Feynman tree diagram which is nonplanar for the ordering 12345 of the external particles, because of the pole in $(p_3+p_5)^2$. The diagram would be planar if the external particles were ordered 12435.

invariants

$$s_{ij}=(p_i+p_{i+1}+\cdots+p_j)^2, \qquad (7.20)$$

but planar tree diagrams do *not* have poles in such *nonplanar* channels as $(p_2+p_5)^2$, $(p_1+p_3+p_4)^2$, or $(p_i+p_{i+2})^2$. To have poles in nonplanar channels we need diagrams such as shown in Fig. 7.5. These diagrams can be made planar by changing the order of external particles. Complete crossing symmetry for a system of 0^+ particles demands singularities in all these channels, and this suggests we make the decomposition

$$T(p_1,\cdots,p_N)=\sum_{\substack{\text{nonequivalent}\\ \text{permutations}\\ (i_1,\ldots i_N)}}^{\frac{1}{2}(N-1)!} B_N(p_{i_1},\cdots,p_{i_N}), \qquad (7.21)$$

where cyclic and anticyclic permutations of the ordering of the particles are considered equivalent. All available channels will be planar in one of the orderings in (7.21), and $T(p_1,\cdots,p_N)$ will be completely crossing symmetric.

In analogy to the case of the four-particle amplitude discussed earlier, we want the residue of a pole in a channel invariant, s_{ij}, to be a polynomial in the *overlapping* variables which are related to the cosine of the scattering angle in that particular channel. The enumeration of these overlapping variables is not completely straightforward for the case of N particles. Basically, overlapping variables are those Mandelstam invariants in which Feynman tree graphs cannot have simultaneous poles. For example, for a four-point function both t and u overlap s. Another definition of overlapping variables involves the use of dual diagrams. These dual diagrams have nothing to do with duality as preached by Dolen, Horn, and Schmid (1968), nor are they the same as the duality diagrams of Harari (1968) and Rosner (1968), with which they share a confusing nomenclature. Rather, these dual diagrams are those discussed, for example, by Eden, *et al.*† (1966) in connection with Feynman diagrams. They are constructed from Feynman diagrams by enclosing each Feynman vertex by a polygon, each side of which is identified with one of the lines entering the vertex. Fig. 7.6 illustrates the one-to-one correspondence of the diagonal lines in these dual diagrams with the Mandelstam invariants. Variables are then said to be overlapping if the diagonals corresponding to them cross.

(ii) *The N-Point Function B_N*

We now turn to the problem of writing a narrow resonance function which has poles in all the channels, (7.20), which are planar for a given ordering of the external particles. Defining the linear trajectory function

$$\alpha_{ij}=a+bs_{ij}, \qquad (7.22)$$

we want a generalization of the integral representation of the beta function

$$B_4(p_1,p_2,p_3,p_4)=\int_0^1 dx x^{-\alpha_{12}-1}(1-x)^{-\alpha_{23}-1}, \qquad (7.23)$$

which we will identify with the 4-particle scattering amplitude (Veneziano, 1968).

In analogy to the situation in the four-particle model, the integral representation for the amplitude, B_N, can be constructed by considering an integration variable, x_{ij}, "conjugate" to the trajectory α_{ij}. We guarantee that there are no simultaneous poles in overlapping variables by requiring

$$x_{ij}=1-\prod_P x_P, \qquad (7.24)$$

where P is the set of channels which overlap s_{ij}. Any set of $N-3$ nonoverlapping variables can be chosen independent. The most convenient set of independent variables corresponds to the poles in the multiperipheral diagram, Fig. 7.7(a):

$$u_j=x_{1j} \qquad (j=2,\cdots,N-2). \qquad (7.25)$$

In terms of the u_j, the solution of the constraint equation (7.24) is given by (Chan and Tsou, 1969)

$$x_{ij}=\frac{(1-u_i\cdots u_{j-1})(1-u_{i-1}\cdots u_j)}{(1-u_{i-1}\cdots u_{j-1})(1-u_i\cdots u_j)} \qquad (7.26)$$

and the N-point function is then given by[38]

$$B_N(p_1\cdots p_N)=\int_0^1\prod_{j=2}^{N-2}du_j(1/J_1)\prod_{i<j}x_{ij}{}^{-\alpha_{ij}-1}, \qquad (7.27)$$

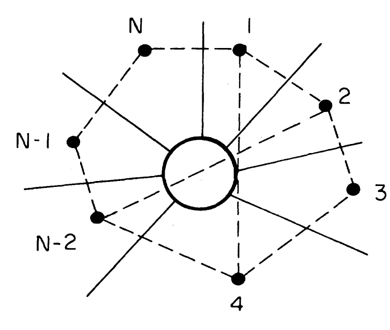

FIG. 7.6. Dual diagram for N-point function. The external lines represent the skeleton. The two diagonal lines represent the channels $s_{14}=(p_1+p_2+p_3+p_4)^2$ and $s_{2,N-2}=(p_2+\cdots+p_{N-2})^2$. Since the diagonals cross, the channels they represent overlap each other. Examples of channels which do not overlap s_{14} include $(p_2+p_3)^2$ and $(p_5+p_6+p_7)^2$.

[38] The singularity structure and asymptotic behavior of B_5 were first discussed by Dixon (1905).

where

$$J_1 = \prod_{i<j} (x_{ij})^{j-i-1}. \tag{7.28}$$

This integral representation of the B_N function clearly indicates that it is invariant under a cyclic or anticyclic permutation of the particle indices and that the only singularities of the function are poles at integral values of Regge trajectories α_{ij}. A convenient form for discussing the other properties of B_N which combines (7.26)–(7.28) is the recursive definition (Bardakci and Ruegg, 1969):

$$B_N(p_1\cdots p_N) = \int_0^1 du_2\cdots du_{N-2} I_N(u_2, \cdots, u_{N-2}), \tag{7.29}$$

where

$$I_N(u_2, \cdots, u_{N-2}) = u_2^{-\alpha_{12}-1}(1-u_2)^{-\alpha_{23}-1}(1-u_2u_3)^{-\Delta_{24}}$$
$$\times\cdots(1-u_2\cdots u_{N-2})^{-\Delta_{2,N-1}} I_{N-1}(u_3, \cdots, u_{N-2}) \tag{7.30}$$

and

$$\Delta_{ij} = (\alpha_{ij}-\alpha_{i+1,j}) - (\alpha_{i,j-1}-\alpha_{i+1,j-1}). \tag{7.31}$$

There is no claim that the form (7.29) is a unique solution to the problem of the N-particle narrow resonance model. The complete problem of uniqueness has not been solved, but, in analogy to the case of the four-particle amplitude discussed in Sec. III, it is always possible to multiply the integrand in (7.29) by an arbitrary function which is symmetric under a cyclic permutation of the channel invariants and well behaved in the region of integration, the pole structure being determined by the end-point properties only.[39]

The $\frac{1}{2}N(N-3)$ channel invariants which appear in (7.29) are not, of course, all independent for $N\geq 6$, but the model is presumed to hold for the physical amplitude when the momentum conservation and mass shell constraints are used to reduce the independent channel invariants to the usual number $3N-10$. As discussed in Sec. V, this feature is of great importance when one tries to construct amplitudes for external currents.

(*iii*) *Properties of* B_N

The Regge asymptotic limit of B_N can be checked with the aid of (7.29). A new ingredient enters when several kinematic variables go to infinity together. This is called the multi-Regge limit (Bali, Chew, and Pignotti, 1967). For example, for B_6 we can consider

$$s_{23}\rightarrow\infty,\ s_{34}\rightarrow\infty,\ s_{45}\rightarrow\infty, \tag{7.32}$$

$$s_{12},\ s_{13},\ s_{14}\quad \text{constants}, \tag{7.33}$$

and

$$s_{23}s_{34}/s_{24} = -\kappa_2, \qquad s_{34}s_{45}/s_{55} = -\kappa_3, \quad \text{constants}. \tag{7.34}$$

[39] Presumably the factorization properties and therefore the degree of degeneracy change rather radically if one changes the integrand in the above fashion. See Frampton (1969b).

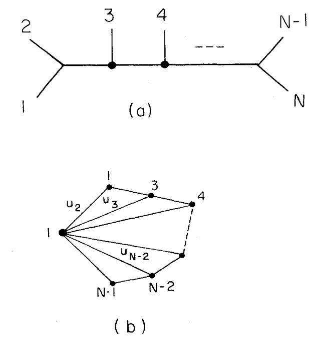

FIG. 7.7(a). Multiperipheral Feynman diagram for an N-particle amplitude. Internal poles occur in the channels s_{12}, s_{13}, $\cdots$, s_{1N-2}. (b). Dual diagram for (a) showing that the channels s_{12}, s_{13}, $\cdots$, s_{1N-2} do not overlap. For each channel, s_{1j}, we introduce the integration variable u_j to define the integral representation of B_N.

Making the change of variables (Bardakci and Ruegg, 1969)

$$1-u_i = \exp(z_i/\alpha_{i,i+1}) \qquad (i=2,4)$$
$$\cong 1+z_i/\alpha_{i,i+1}, \tag{7.35}$$

taking the high-energy limit under the integral sign in (7.29) in the form

$$\lim_{\alpha\rightarrow\infty} (1+\sigma/\alpha)^\alpha = \exp(\sigma), \tag{7.36}$$

and making use of the high-energy approximation $\Delta_{ij}\approx\alpha_{ij}$, we get

$$B_6 \sim (-\alpha_{23})^{\alpha_{12}}(-\alpha_{34})^{\alpha_{13}}(-\alpha_{45})^{\alpha_{14}}$$
$$\times\int_0^\infty dz_2dz_3\, dz_4 z_2^{-\alpha_{12}-1} z_3^{-\alpha_{13}-1} z_4^{-\alpha_{14}-1}$$
$$\cdot \exp\left[-\left(z_2+z_3+\frac{z_2z_3}{\kappa_2}+\frac{z_3z_4}{\kappa_3}+\frac{z_2z_3z_4}{\kappa_2\kappa_3}\right)\right]. \tag{7.37}$$

We can also see that the contribution of each trajectory to the multi-Regge limit factorizes separately by writing (7.37) in the form

$$B_6 \sim \Gamma(\alpha_{12})[-\alpha_{23}]^{\alpha_{12}} V(\alpha_{12}, \alpha_{13}, \kappa_2)\Gamma(-\alpha_{13})[-\alpha_{34}]^{\alpha_{13}}$$
$$\cdot V(\alpha_{13}, \alpha_{14}, \kappa_3)\Gamma(-\alpha_{14})[-\alpha_{45}]^{\alpha_{14}}. \tag{7.38}$$

The function $V(\alpha, \alpha', \kappa)$, which can be identified with the vertex for two Regge trajectories (Reggeons) and an external pole, is then

$$V(\alpha, \alpha', \kappa) = [\Gamma(-\alpha)]^{-1}[\Gamma(-\alpha')]^{-1}$$
$$\times\int_0^\infty dz_1\, dz_2 z_1^{-\alpha-1} z_2^{-\alpha'-1} \exp\left\{-\left[z_1+z_2+\frac{z_1z_2}{\kappa}\right]\right\}. \tag{7.39}$$

The variable $(-1/\kappa)$ in this vertex function is identified with the Toller variable (Toller, 1965) of Bali, Chew, and Pignotti (1967). Dependence upon the Toller

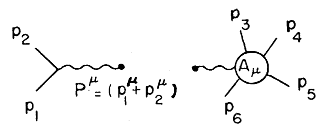

FIG. 7.8. An illustration of the process of finding the pole at $\alpha_{12}=1$ in B_6 in order to construct the amplitude $P^\mu A_\mu(P, p_3, p_4, p_5, p_6)$ for the coupling of a spin one particle to four scalars.

variable was often assumed to be absent in early studies of the multi-Regge limit and the Veneziano model was among the first to predict a definite dependence on this variable. This dependence has not been verified phenomenologically as yet (Barshay, 1969; Tan and Wang, 1969).

The multi-Regge behavior of B_N given by (7.37) does not necessarily guarantee that the complete amplitude, (7.21), which is a sum over B_N's, will have the proper Regge behavior. We must also show that B_N is exponentially decreasing when we fix a nonplanar channel invariant in which it has no poles, just as was the case for an exotic channel in the four-point amplitude in Sec. III. For general N, the nature of the constraints which reduce the number of variables from $\frac{1}{2}N(N-3)$ to $3N-10$ makes proof of this property difficult and we are not aware of the existence of any such proof. See the discussions of Białas and Pokorski (1969) and of Zakrzewski (1969).

(iv) *Factorization and Projections*

Factorization of the leading trajectory ensures that

$$\lim_{\alpha_{1j}\to 0} \alpha_{1j} B_N(p_1, \cdots, p_N) = B_{j+1}(p_1\cdots p_j, p_I) B_{N-j}(p_I, p_j\cdots p_N), \quad (7.40)$$

which can be verified by noting that, in the limit $u_j \to 0$, the integrand in (7.29) separates into the appropriate factors. We can also project out couplings to resonances of nonzero angular momentum on the parent Regge trajectory in order to define amplitudes for particles with spin. (See Bardakci and Halpern, 1969; Campbell, Olive, and Zakrzewski, 1969.) For example, in the 6-point function, the residue of the pole at $\alpha_{12}=1$ can be used to construct the invariant amplitude for the coupling of the spin-1 particle on the parent trajectory to four scalars. Using the notation of Fig. 7.8 with $P^\mu = p_1^\mu + p_2^\mu$ and $\alpha_{p3} = a+b(P+p_3)^2$, $\alpha_{p4} = a+b(P+p_3+p_4)^2$, we write

$$\begin{aligned} P^\mu A_\mu(P, p_3, p_4, p_5, p_6) &= 2bP^\mu[p_{3\mu}B_5(\alpha_{p3}, \alpha_{p4}, \alpha_{34}, \alpha_{45}, -2b(p_3\cdot p_5)) \\ &+ p_{4\mu}B_5(\alpha_{p3}-1, \alpha_{p4}, \alpha_{34}, \alpha_{45}, -2b(p_3\cdot p_5)) \\ &- p_{5\mu}B_5(\alpha_{p3}-1, \alpha_{p4}-1, \alpha_{34}, \alpha_{45}, -2b(p_3\cdot p_5))], \end{aligned} \quad (7.41)$$

where B_5 is the five-point amplitude of (7.29) with the order of the variables given by that integrand and with certain trajectories lowered as indicated. More complicated spin projections can be made with the aid of the series representation for B_N found by Hopkinson and Plahte (1969). For example, the series form for B_5

$$B_5(p_1\cdots p_5) = \sum_{k=0}^{\infty} (-1)^k \binom{z_{24}}{k} B_4(\alpha_{34}, \alpha_{45}+k) B_4(\alpha_{12}+k, \alpha_{23}), \quad (7.42)$$

where $z_{24} = \alpha_{24} - \alpha_{34} - \alpha_{23}$, can be used in conjunction with the expansion of the beta function to yield

$$B_5(p_1\cdots p_5) = \sum_{k=0}^{\infty}\sum_{L=0}^{\infty} (-1)^{k+L} \binom{z_{24}}{k}\binom{\alpha_{23}}{L} \times \frac{B_4(\alpha_{34}, \alpha_{45}+k)}{-\alpha_{12}+k+L}. \quad (7.43)$$

This illustrates explicitly the Feynman diagram structure and the couplings of the model and can be used to verify the generalization of atonous duality (Sec. II) appropriate for the five-particle form. These considerations of factorization are relevant to the problem of uniqueness. Although (7.40) provides some constraint, there still exists the possibility of multiplying the integrand in (7.29) by a function which preserves the factorization property. The class of such functions has been discussed by Gross (1969).

So far we have just discussed factorization in terms of leading behavior and poles on the leading trajectory. Since factorization can be thought of as a form of single-particle unitarity which can lead, in principle, to important constraints, it is interesting to try to extend the factorization property to particles lying on daughter trajectories.

At the daughter level, simple factorization does not exist. Instead, it has been shown by Bardakci and Mandelstam (1969) and by Fubini and Veneziano (1969) that, in order to preserve factorization, the lower levels must become degenerate so that the residue of a pole can be expressed as a finite sum of factors which does not depend on the number of external lines. Because of the cyclic symmetry of B_N, we need only establish this property for one channel, s_{ij}. Looking at the integrand in (7.29), we see that only

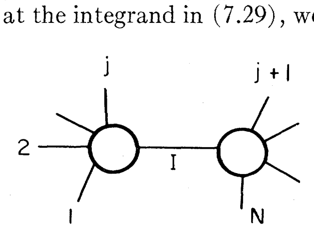

FIG. 7.9. The residue of a pole at $a+b(p_1+\cdots p_j)^2=n$ can be represented as a sum of terms $\sum_k C_k(p_1\cdots p_j)\cdot D_k(p_{j+1}\cdots p_N)$ where C_k depends only on $p_1, \cdots, p_j$ and D_k on $p_{j+1}, \cdots, p_N$. As can be seen from the diagram, we are restricting attention to only one ordering of external particles in making this decomposition.

a certain number of factors contain the variable u_j. We make the constraint $i<j<k$ and let $I_1(u_k, p_i)$ be those factors in (7.29) which involve only variables in the left half of Fig. 7.9 and $I_2(u_k, p_k)$ be those factors which contain only variables on the right half. If we then lump all those factors which have any u_j dependence into $F(u_j; u_i, p_i; u_k, p_k)$, we see that we can write the residue of the pole in (7.29) at $\alpha_{1j}=n$ as

$$R=\int_0^1 \prod_{i<j<k} du_i\, du_k I_1(u_i, p_i)$$

$$\times I_2(u_k, p_k)(n!)^{-1}\frac{\partial^n}{\partial w^n} F(w; u_ip_i, u_kp_k)\mid_{w=0}. \quad (7.44)$$

The function, F, can be written in the form

$$F=\exp\left\{\sum_{r=1}^{\infty}\frac{w^r}{r}[P_1^{\mu}(p_i, u_i, r)P_2^{\mu}(p_k, u_k, r)+c(r)]\right\}, \quad (7.45)$$

where c is a constant and $P_1^{\mu}(P_2^{\mu})$ is a four-vector depending only on the variables p_i and $u_i(p_k$ and $u_k)$. The derivation in (7.44) can be verified by expanding F in a power series in w and isolating the term with power w^n. The number of factors depends only on the functional form of F and not on the specific form of $P_1(P_i, u_i, r)$ and, therefore, does not depend on the number of external lines. The number of different factors can be shown to equal the number of ways of choosing nonnegative integers f_m to satisfy the partition equation

$$\sum_{m=1}^{\infty} m\cdot f_m=n. \quad (7.46)$$

For large n this number, d_n, increases approximately as

$$d_n\sim \exp[2\pi(n/6)^{1/2}], \quad (7.47)$$

so that the level degeneracy increases exponentially with mass.[40] Curiously enough, this is the same sort of structure predicted by Hagedorn (1968) on the basis of a thermodynamic model which treats hadrons as bound states of each other, in terms of a statistical ensemble depending on a universal temperature. It is interesting that two very different models which lay claims, however tenuous, to being bootstrap models should predict the same sort of degeneracy in the hadron spectrum (see Krzywicki, 1969). There remains the problem of exposure of this prediction to experiment. However crowded the experimental hadron spectrum may seem (Rosenfeld *et al.*,* 1969), there is as yet no evidence for this sort of multiple structure. If this type of degeneracy really existed, resonances would have decay modes whose properties depended on their production mechanism, and we are not aware of experimental evidence for the lack of simple factorizability of any known resonance. These questions can be circumvented in two ways. One can claim that experiments have not yet probed energies at which these features become prominent. It then becomes necessary to revamp the basic notion of what a particle is. Alternatively, we can ignore the properties predicted by the model for lower trajectories on the ground that these trajectories are mimicking the effect of background in a true amplitude. Since many of the lower trajectories have negative residues, thus providing a set of ghost states entirely distinct from those associated with the spin structure discussed in Sec. VII.B, the latter approach temporarily avoids confrontation with the difficulties associated with ghosts, at least those of this second kind. Neither of these escapes appeals to the authors, and in fact we see no reason to believe the detailed resonance spectrum of the model either at the parent or the daughter level.[41]

There exist identities, called "Ward identities" by Fubini and Veneziano (1969), which indicate that certain linear combinations of the states counted in (7.46) correspond to functions which are total derivatives of one of the u_i or u_k so that the integration in (7.44) causes the contribution of these states to vanish. The total reduction in the degeneracy, (7.47), for large n caused by these identities is negligible, but they can be used to show, for instance, that some states which have negative residues (ghosts) are compensated by similar poles with positive residues (Fubini and Veneziano, 1969; Bardakci and Mandelstam, 1969).

The level structure of the Beta function form has been studied extensively in terms of a harmonic

[40] The quantity d_n is called the partition number by Fubini and Veneziano (1969). See their paper for further details.

[41] The second set of ghosts in the orbital factor and its association with an indefinite metric are best seen using the oscillator-operator formalism of Fubini, Gordon, and Veneziano (1969). The problems associated with these ghosts are presumably identical to those encountered by Lee (1969) and Lee and Wick (1969). In terms of the operator formalism, the field operators satisfy the indefinite metric commutation relations

$$[a_{\mu}^{(n)},\quad a_{\nu}^{\dagger(m)}]=\delta_{nm}g_{\mu\nu}.$$

The timelike oscillator operators therefore create infinities of both normal and ghost states. In terms of these harmonic oscillator operators, the total energy is given by

$$H=-\sum_{n=1}^{\infty} na^{\dagger(n)}a^{(n)}.$$

If, as before, we define

$$\alpha_K=a_K+b(\sum_{i=1}^{\infty} p_i)^2,$$

we can write

$$B_N(p_i, \cdots, p_N)=\langle 0\mid V(p_{N-1})D(\alpha_{N-2})\cdots D(\alpha_2)V(p_2)\mid 0\rangle,$$

where

$$D(\alpha)=\int_0^1 dx x^{-\alpha+H-1}(1-x)^{a-1}$$

and

$$V(p_K)=\exp\left\{-\sum_{n=1}^{\infty}\left(\frac{2b}{n}\right)^{1/2}a^{\dagger(n)}p_K\right\}\exp\left\{\sum_{n=1}^{\infty}\left(\frac{2b}{n}\right)^{1/2}a^{(n)}p_K\right\}.$$

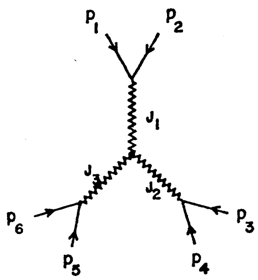

FIG. 7.10. Vertex function for coupling of particles on leading trajectories having spins J_1, J_2, and J_3, as computed using B_6.

oscillator model, (Nambu, 1969; Susskind, 1969, 1970), and an operator formalism has been developed by Fubini, Gordon, and Veneziano (1969).[41] These studies are suggestive in that they indicate a connection between the model and infinite-component field theories and they may provide a connection between the negative residue states and other features, such as asymptotic behavior of the model. In particular, the operator formalism can be used to isolate vertices between the factorized excited particles in the model (Sciuto, 1969; Stapp, 1970). It can also be used to invent twisting operators (Amati, Bellac, and Olive, 1969; Caneschi, Schwimmer, and Veneziano, 1969) which can be used to define the signature of internal states. (See also the discussion on signature by Hopkinson and Chan, 1969; Zakrzewski, 1969.)

As discussed by Gross (1969), these factorization considerations have some bearing on the problem of the uniqueness of the B_N. The number of levels increases, in general, when the integrand in (7.29) is multiplied by a function $f(u_2, \cdots, u_{N-3})$, but it remains finite for a large class of functions. This suggests that the level structure, given by (7.45), is in some sense minimal (Olesen, 1970a), but the situation is not completely clear.

(v) *An Important Simplification*

For many applications, the full form of B_N is unnecessarily complicated and it is desirable to use an approximation invented by Bardakci and Ruegg (1969) in which all factors in the integrand in (7.29) containing more than two u_j's are omitted:

$$(1-u_iu_{i+1}u_{i+2}\cdots u_{i+n})\rightarrow 1 \text{ in } (7.29). \qquad (7.48)$$

For large N this can simplify the integrand in (7.29) considerably. For example, in this approximation we would write the six-point function as

$$B_6 \cong \int du_2du_3\, du_4 u_2^{-\alpha_{12}-1}u_3^{-\alpha_{13}-1}u_4^{-\alpha_{14}-1} \times(1-u_2)^{-\alpha_{23}-1}(1-u_3)^{-\alpha_{34}-1}(1-u_4)^{-\alpha_{45}-1} \times(1-u_2u_3)^{-2b(p_2\cdot p_4)+a+bm^2}(1-u_3u_4)^{-2b(p_3\cdot p_5)+a+bm^2}, \qquad (7.49)$$

which is a good approximation of the original six-point function in the multi-Regge limit, (7.37), and can be used in schemes based upon the multiperipheral concept as discussed by Chew and Pignotti (1968) and Chew, Goldberger, and Low (1968).

Application to the bootstrap problem

To finish the bootstrap problem, we will use B_{2N} in (7.1) as the orbital factor. From what we have said, this is a self-consistent choice in the sense of factorization, provided one is willing to live with the exponential degeneracy of the lower levels.

To form a four-point function in the bootstrap we need to consider A_8, the four-quark–four-antiquark amplitude pictured in Fig. 7.9. Rather than repeating the details of previous arguments, we will very briefly show how A_8 breaks up, following Bardakci and Halpern (1969).

The spin and $SU(3)$ parts of A_8 factorize easily, as seen from Eqs. (7.6) and (7.16), and we focus on the orbital factor B_8. Suppose we check that B_8 breaks up as in the tree diagram Fig. 7.9. As we have discussed above, there are problems associated with the factorization of the lower trajectories. Let us examine the leading internal trajectory. Then it can be shown that

$$B_8\rightarrow \sum_J [1/(J-\alpha_0)]B_4^{\mu(J)}(p_1, p_2, p_3, p_4) \otimes B_4^{\mu(J)}(p_5, p_6, p_7, p_8), \qquad (7.50)$$

where $\rightarrow$ indicates (7.50) holds only for the highest internal trajectory, $\alpha_0=\alpha[(p_1+p_2+p_3+p_4)^2]$, and $B_4^{\mu(J)}$ is the amplitude for the coupling of four scalar particles to a spin J object. The symbol $\otimes$ indicates that the J indices of the $B_4^{\mu(J)}$, represented by $\mu(J)$, have to be dotted into each other. For additional details the reader is referred to Bardakci and Halpern (1969).

For reference, we give the form of the triple-Regge vertex obtained from B_6 by Misheloff (1969). Let the momenta be as shown in Fig. 7.10, with

$$K_A=\alpha_{35}\alpha_{51}/\alpha_{61}, \qquad K_B=\alpha_{13}\alpha_{35}/\alpha_{45}, \qquad K_C=\alpha_{51}\alpha_{13}/\alpha_{23}. \qquad (7.51)$$

The asymptotic form of B_6 as $|s_{51}|, |s_{35}|, |s_{13}|\rightarrow\infty$ with K_A, K_B, and K_C fixed is

$$B_6\rightarrow(-\alpha_{51})^{\alpha_{12}}(-\alpha_{35})^{\alpha_{56}}(-\alpha_{13})^{\alpha_{34}} \times G(s_{12}, s_{34}, s_{56}; K_A, K_B, K_C), \qquad (7.52)$$

where

$$G(s_{12}, s_{34}, s_{56}; K_A, K_B, K_C) = \int_0^\infty dv_1dv_2\, dv_3 v_1^{-\alpha_{12}-1}v_2^{-\alpha_{56}-1}v_3^{-\alpha_{34}-1} \times \exp\left[-v_1-v_2-v_3+\frac{v_1v_2}{K_A}+\frac{v_2v_3}{K_B}+\frac{v_3v_1}{K_C}\right]. \qquad (7.53)$$

Additional material relevant to this section can be found in Amati, Bellac, and Olive (1969), Barshay (1969), Delbourgo and Rotelli (1969), Freund (1969b), Frye (1970), Gallardo and Susskind (1970), Kugler and Milgrom (1969), Landshoff and Zakrewski (1969), and Susskind (1970).

VIII. NARROW RESONANCES, THE POMERANCHON, EXACT AND BROKEN DUALITY, AND EXOTIC RESONANCES

In this section we will discuss the absence of high-energy elastic diffraction phenomena in dual, narrow resonance models, the consequences of exact duality, and the presence of exotic resonances.

A. The Pomeranchon and Exotic Resonances

Our narrow resonance amplitude, in its present form, cannot describe elastic high-energy diffraction scattering. We will show the reason for this using the example of $\pi\pi$ scattering. Traditionally (Chew and Frautschi, 1961; Frautschi, Gell-Mann, and Zachariasen, 1962), these phenomena have been associated with a Regge trajectory called the Pomeranchon. However, as has been recently pointed out with increasing frequency, it is not at all clear from the available data that such a description is appropriate (see Trilling, 1970). According to the Pomeranchuk conjectures (Pomeranchuk, 1956; Pomeranchuk and Okun', 1956; Pomeranchuk, 1958), at high energy, elastic cross sections are supposed to become independent of isospin.

For the $\pi\pi$ isospin amplitudes, (3.8), this means that

$$X^s(s,t) \underset{s\to\infty,t=0}{\sim} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} f(s). \tag{8.1}$$

Because of our choice of $SU(2)$ solution, with no $I=2$ poles, (8.1) does not hold in the model described in Sec. III; instead we have

$$X^s \underset{s\to\infty,t=0}{\sim} \begin{pmatrix} 3/2(1+e^{i\pi a}) \\ (e^{i\pi a}-1) \\ 0 \end{pmatrix} \Gamma(1-a)(bs)^a. \tag{8.2}$$

To see what is happening, consider the usual invariant amplitude decomposition, (3.1), for the full $\pi\pi$ amplitude. The isospin amplitudes are given by

$$X^s = \begin{pmatrix} A_s^0 \\ A_s^1 \\ A_s^2 \end{pmatrix} = \begin{pmatrix} 3A+B+C \\ B-C \\ B+C \end{pmatrix}. \tag{8.3}$$

In order to have (8.1), we need

$$\lim_{s\to\infty,t=0} [A/B=0, C/B=0]. \tag{8.4a}$$

To make this result independent of channel we also require

$$\lim_{u\to\infty,s=0} [B/A=0, C/A=0], \tag{8.4b}$$

$$\lim_{t\to\infty,u=0} [A/C=0, B/C=0]. \tag{8.4c}$$

Focusing on the s-channel, we can see what the trouble with implementing (8.4) for the narrow resonance model is. The invariant amplitude $B(s,t,u)$ is symmetric in $s\leftrightarrow u$ so that

$$X^u \underset{u\to\infty,t=0}{\longrightarrow} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} f(u) \tag{8.5a}$$

and similarly

$$X^s \underset{s\to\infty,u=0}{\sim} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} f(s), \tag{8.5b}$$

$$X^t \underset{t\to\infty,s=0}{\sim} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} f(t). \tag{8.5c}$$

In the s channel, B dominates in the forward direction and C in the backward direction. We can draw a Mandelstam diagram with the asymptotic behavior for the B amplitude superimposed as in Fig. 8.1.

Now, suppose we want to make a narrow resonance model for $\pi\pi\to\pi\pi$ with no $I=2$ poles and with the Pomeranchon included as an ordinary Regge trajectory, so that

$$X^s \underset{s\to\infty,\text{fixed } t}{\sim} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} s^{\alpha_P(t)}\beta_P(t) \left[\frac{1+e^{i\pi\alpha_P(t)}}{\sin \pi\alpha_P(t)}\right]. \tag{8.6}$$

According to the narrow resonance rules, the directions marked P in Fig. 8.1 are to be associated with t-channel poles, while by excluding $I=2$ poles we know B has no s- or u-channel singularities and the directions marked "0" in Fig. 8.1 therefore indicate an exponential falloff.

No such meromorphic function exists. In order to get the Regge behavior, (8.6), we must have an infinite number of poles in either the s or u channels, as discussed in Sec. II, or we must go beyond narrow resonances and introduce cuts (Jengo, 1969).

Therefore, in order to have a narrow resonance Pomeranchuk trajectory, we must allow $I=2$ poles. This has been suggested by Wong (1969a), who also pointed out that the $I=2$ trajectory may have a large negative intercept, so that the exotic poles $(I=2)$

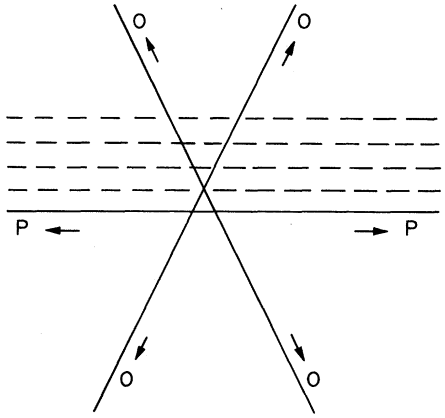

Fig. 8.1. Asymptotic behavior of the invariant amplitude $B(s, t, u)$, defined as in (3.1), for $\pi\pi$ scattering, as required if the Pomeranchuk limit (8.1) is to be satisfied.

appear at arbitrarily high mass. We do not find Wong's suggestion particularly attractive, because it does not seem to solve any problems associated with the Pomeranchon in classical Regge pole treatments (Berger and Fox, 1969; Fox, 1969; Jackson,* 1970; Trilling, 1970). We will discuss this further in Sec. X.

B. Exact Duality

As pointed out above, the narrow resonance model satisfies atonous duality, in the sense that the infinite sum of poles in one channel diverges to produce cross channel poles. This form of exact duality, in which amplitudes can be represented as Reggeized sums of Feynman trees, necessitates the presence of "exotic" resonances, in certain baryon–antibaryon annihilation channels. (We *define* the following as exotic: mesons outside **1** or **8** in $SU(3)$; baryons not belonging to **1**, **8**, or **10**, or having baryon number larger than one.) The necessity for the appearance of exotics was first pointed out by Rosner (1968).

We will discuss below meson–meson, meson–baryon, and baryon–baryon scattering, and will indicate the general form that self-consistent solutions to the $SU(3)$ crossing problem must take. Readers interested in more details are referred to the work of Rosner (1969b), Rosner, Rebbi, and Slansky (1969), and Mandula *et al.* (1969).

The most elegant way to see that exotics are needed is to use the duality diagrams[42] of Rosner (1969a) and Harari (1969), which are pictorial ways of writing $SU(3)$ crossing matrices for N-point functions whose legs transform like **3** or **3*** under $SU(3)$. In the usual language, each line in a duality diagram represents an ace-quark (Gell-Mann, 1964b; Zweig, 1964), and if we look at $2N$ point functions having N external quarks and N external antiquarks, we can decide, for N meson scattering, which eigenvectors of the $SU(3)$ crossing matrices with eigenvalue one, are allowed.[43]

In terms of the discussion of Sec. VII, it is always possible to write an N-point function as the sum over products of an internal symmetry factor and an ordinary space factor. In VII we went even farther and factored the ordinary spin piece into orbital and spin terms. This last factorization is not required by any physical principle, and probably has nothing to do with reality. From the duality diagram point of view we consider the internal symmetry factor only, and this is perfectly legitimate.

The quarks in duality diagrams therefore have $SU(3)$ quantum numbers only. Mesons are formed from quark–antiquark pairs and appear in nonets $[\mathbf{3}\otimes\mathbf{3}^*=\mathbf{1}\oplus\mathbf{8}]$. Baryons are formed from quark triplets $[\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}'\oplus\mathbf{10}]$. In Fig. 8.2(a) we show duality diagrams for the scattering of meson nonets $M+M\rightarrow M+M$. As can be seen in Fig. 8.2(a), a nonet eigenvector of the $SU(3)$ crossing matrix exists, having $Q\bar{Q}$ pairs intermediate in every channel. In other words, it is possible, from the internal symmetry viewpoint, to build a completely self-consistent narrow resonance world out of meson nonets alone, and in fact this is what we discussed in Sec. VII.

For meson–baryon scattering, $MB\rightarrow MB$ in the s and u channels, $M\bar{M}\rightarrow B\bar{B}$ in the t channel, we want a solution with $[\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}'\oplus\mathbf{10}]$ in s and u, $[\mathbf{1}\oplus\mathbf{8}]$ in t. This also exists (Roy and Suzuki, 1969; Mandula *et al.*, 1969), and so far exotics are not required. Duality

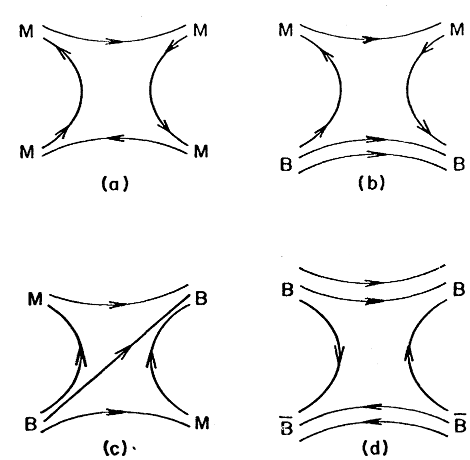

Fig. 8.2. Duality diagrams for (a) meson–meson scattering with nonets in both channels; (b) meson–baryon scattering with baryon exchange in one channel and meson exchange in the other; (c) meson–baryon scattering with baryon exchanges in both channels; (d) baryon–baryon scattering showing meson exchange in one annihilation channel and exotic $qq\bar{q}\bar{q}$ exchange in the other.

[42] We emphasize that these duality diagrams are *not* the dual diagrams connected with the singularity structure of Feynman diagrams.

[43] Caution is necessary in interpreting duality diagrams so as to give definite results regarding $SU(3)$ crossing matrices. For example, straightforward symmetrization of quark lines can easily lead to incorrect conclusions. We are indebted to J. Mandula for pointing this out to us.

diagrams for the meson–baryon process are shown in Fig. 8.2(b), (c).

Consider now baryon–baryon scattering, $BB \to BB$ in the t channel, $B\bar{B} \to B\bar{B}$ in the s and u channels. If resonances appear in the t channel, they will be exotic ones with baryon number $B=2$. There is as yet no definitive experimental evidence for the existence of exotics and we therefore may want to eliminate such objects.[44] For the scattering of octets ($O\bar{O} \to O\bar{O}$ in t, etc.) this is possible to do without reaching any contradiction with the $MM \to MM$ and $MB \to MB$ amplitudes. Trouble, however, arises when we try to force out exotics in the octet–decuplet ($OD \to OD$) and decuplet–decuplet ($DD \to DD$) channels (Rosner, 1968; Roy and Suzuki, 1969; Rosner, Rebbi, and Slansky, 1969). The absence of $B=2$ exotics in the decuplet processes results in exotic mesons in the $O\bar{D} \to O\bar{D}$ and $D\bar{D} \to D\bar{D}$ channels.

One can reach this result either by drawing duality diagrams for $BB \to BB$, as in Fig. 8.2(d), or by directly using the $SU(3)$ crossing matrices (Rebbi and Slansky, 1969).

Roughly, what is happening here is as follows. The relevant Clebsch–Gordan series for the processes with decuplets are (de Swart, 1963)

$$\mathbf{8} \otimes \mathbf{10}^* = \mathbf{8} \oplus \mathbf{10}^* \oplus \mathbf{27} \oplus \mathbf{35}^*,$$

$$\mathbf{10} \otimes \mathbf{10}^* = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{27} \oplus \mathbf{64}.$$

Except for very peculiar circumstances—such as when scattering self-adjoint representations like **8**'s—the crossing matrices will not be diagonal in a particular single representation.[45] One expects to have more than one representation appearing as intermediate states in a given channel. This makes it very difficult to eliminate exotics if, as in **8**⊗**10*** above, there is only one normal representation available. Either **10*** or **27** must occur (Rosner, Rebbi, and Slansky, 1969). In **10**⊗**10*** it turns out that the **1** does not help and that there is no way to eliminate **27**.

There is always the escape from this situation considered in the $\pi\pi$ case above. We can try to force the exotics to appear at very high mass, by making the intercept of the leading exotic trajectory large and negative. This may very well be how nature works. We have been unable to construct an argument which eliminates this possibility, in the narrow resonance framework, though we strongly suspect that one exists and that exotic objects appear as consequences of nonlinear unitarity constraints. We can only say that in the present state of the model (a) there is no way of telling at what level exotics appear, and (b) the introduction of exotics solves none of the difficulties of classical Regge pole phenomenology.

In terms of duality diagrams exotics appear in baryon–baryon scattering because it is topologically impossible to construct a baryon–baryon picture having only $Q\bar{Q}$ in the s and u channels. States containing $Q\bar{Q}\,Q\bar{Q}$ must appear. Freund, Waltz, and Rosner (1969) have suggested a selection rule, constructed along the lines of a previous suggestion of Lipkin (1966), which limits exotic states to $Q\bar{Q}\,Q\bar{Q}$ resonances coupling to $B\bar{B}$ systems. We are skeptical that nature has chosen such an arbitrary construction.

Mandelstam (1970a) has suggested that exotic states can be incorporated into the narrow resonance model by making the intercepts of trajectories a quadratic function of the total quark number (quark plus antiquark number). This increases the degeneracy of the level structure when we try to factorize the pole residues, as discussed in Sec. VII (Olesen, 1970a). The degeneracy is still exponential, but, in mass terms of the analogy with Hagedorn's statistical model (Hagedorn, 1968), the temperature is higher so that

$$\underset{\text{one } 0+ \text{ trajectory}}{\exp\,(4.1s^{1/2})} \to \underset{\text{exotic trajectories}}{\exp\,(6.2s^{1/2})}\ . \qquad (8.7)$$

For the reasons given in Sec. VII, and to be discussed further in XI, we also find Mandelstam's procedure unattractive.

There are additional undesirable predictions of exact duality that we have already touched on in Sec. VII. As we saw there, certain unobserved trajectories arise. For example, from $\rho\rho$ or $\rho\pi$ scattering one deduces the π trajectory must have an $I=0, J^{PC}=0^{--}$ degenerate partner.

Because of its various disabilities outlined above, we believe that duality must be broken, and that this breaking will be associated with high-energy elastic diffraction, the existence of baryons, and the nonexistence of leading exotics. In the next section we will discuss some ways in which duality breaking could occur.

C. Breaking Duality

For the reasons discussed above, we believe duality must be badly broken in nature. We will discuss here an interesting semiquantitative suggestion of how this comes about [due to Mandula, Weyers, and Zweig (1969a)].

As pointed out by Schmid and Yellin (1969), in order for the FESR bootstrap, defined in Sec. II, to work, the narrow resonance approximation and the parameterization of the amplitude with Regge poles must have overlapping regions of validity. Resonance saturation requires small s (or cutoff N), while the Regge assumption requires s large. In order for the scheme to work, we require s to be in the interval $s_1 \leq s \leq s_2$, where s_1 is the *minimum* value for which the Reggeization is

[44] There is some recent experimental evidence in favor of the existence of exotics. However, this evidence is inconclusive. See the review of Tripp* (1969), Sec. VI, and also Kato *et al.* (1969).

[45] In fact, it is quite possible that, except in trivial cases, it can be shown that crossing matrices for an arbitrary Lie algebra never have only the diagonal element nonzero in a particular row and column pair.

good, and s_2 marks the *maximum* value of s for which narrow resonances saturate the FESR.

Mandula *et al.* hypothesize that s_1 should be associated with the position of the threshold in question, while s_2 is related to the point at which inelasticity sets in. Fixing s_2 in this way follows from the notion that the nonresonant background in the FESR represents the contribution of the Pomeranchon (Freund, 1968a; Harari, 1968) and that the Pomeranchon arises from the presence of the infinity of inelastic channels. As far as s_1 goes, it is by no means selfevident that the Regge series fails to make sense below threshold, but if we accept the Mandula *et al.* guess that it does not, we have a partial explanation for the difficulties with $\bar{B}B$ channels discussed above. In $\bar{B}B \to MM$, using these ideas, $s_1 \gtrsim 4M_B^2$, and s_2 is probably near 1 BeV^2, so that there may be no overlap at all, and duality is maximally broken. Once one accepts these ideas, it is straightforward to write down a hierarchy of reactions in which duality is more and more badly broken.[46] One can even try to make some rough comparison with experiment. See Mandula *et al.* for the details of such an effort.

We have been rather cavalier above about ignoring exchange degeneracy, and concentrating on $SU(3)$ quantum numbers only. For example, in the meson–meson and meson–baryon cases the usual $J^{PC}=1^{--}(\rho)$ and $J^{PC}=0^{-+}(\pi)$ trajectories must be accompanied by $2^{++}(f, A_2)$ and $1^{+-}(B)$ partners in order to achieve self-consistency. Further details regarding this can be found in the discussions of Mandula *et al.* (1969a, b) and of Rosner, Rebbi, and Slansky (1969).

Additional work relevant to this section can be found in Kato *et al.* (1970), Neville (1969), Schmid (1969b), Schwimmer (1969), and Yellin (1969c). For an extensive review of the subjects of this section see Mandula, Weyers, and Zweig (1970).

IX. CLOSED LOOPS—REGGEIZED PERTURBATION SERIES

An attempt has been made to generate a theory which takes the tree diagrams present in the simple Veneziano model and uses them as Born terms in a perturbation series. This approach is motivated by the factorization properties of the Veneziano model discussed in Sec. VII and a bit of field theory folklore commonly known as the "tree theorem." Briefly, the tree theorem states that, in a perturbation theory with factorized pole residues, unitarity sums which involve a complete set of two particle intermediate states can be performed by combining two external legs of a tree diagram to form a loop (see Fig. 9.1). Only one set of intermediate states need be summed. A common terminology is that the loop is "sewn together" from the tree diagram (Bardakci, Halpern, and Shapiro, 1969). If it converges, a perturbation series based on such factorizable loops will produce a unitary, though not necessarily correct, S matrix.[47]

In this section, we would like to discuss the construction of a simple square graph from the N-particle narrow resonance amplitude and examine some of the difficulties which occur when we try to enforce factorization at the daughter level. We would also like to mention briefly the construction of diagrams with twisted loops and examine some of their properties.

Reggeized, dual, closed-loop diagrams suffer from the same maladies which affect the N-point functions discussed in Sec. VII. Furthermore, as we shall see below, in trying to form an amplitude with closed loops, the loop integrand itself, which is defined by an infinite product, diverges at one corner of the integration volume. Mandelstam (1970b) has suggested this difficulty arises because of the exponential degeneracy of lower trajectories discussed in Sec. VII. This points up again what we have emphasized repeatedly above. Each additional requirement imposed on narrow resonance amplitudes thus far has led to further complications.

A. Construction of the Square Graph

To illustrate the techniques involved in the construction of functions with internal loops, we will form the integral representation of an amplitude with the singularity structure of a simple square graph. In analogy with the approach discussed in Sec. VII, we associate with each of the internal lines in Fig. 9.2 an integration variable $u_j(j=1, 2, 3, 4)$ and tentatively

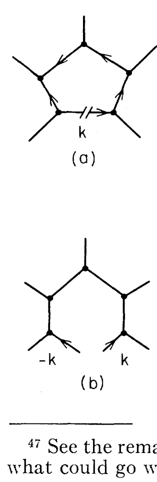

Fig. 9.1. Factorization relates a diagram such as (a) to a diagram such as (b), where the two cut legs represent identical particles with opposite momenta.

[46] Mandula *et al.* (1969a, b) obtain the hierarchy mentioned above and diverse other results by reducing the problem to a set of bilinear constraints on coupling constants similar to those obtainable by using the N/D method (Cutkosky, 1963), a $Z=0$ field theory (Kaus and Zachariasen, 1968), or a straight narrow resonance model such as that discussed in the present work.

These constraints suffer from the same limitations discussed in Sec. II.A and do not possess a unique solution. Mandula *et al.* make a very clever choice of solution in order to obtain gross agreement with experiment. [For alternate solutions, see Capps (1969).] Since these relations could equally well be obtained by using the assumptions outlined in Sec. II.A, the justification of these results awaits the construction of a model which does not suffer the grievous pathologies of the narrow resonance model discussed in the text.

[47] See the remarks at the end of Sec. IX.C for speculation about what could go wrong with the perturbation series.

write the amplitude in the form

$$S(q_1\cdots q_4)=-g^4\int d^4k$$

$$\times\int_0^1 du_1du_2du_3du_4u_1^{-a-1-b(k+q_1)^2}u_2^{-a-1-b(k+q_1+q_2)^2}$$

$$\times u_3^{-a-1-b(k-q_4)^2}u_4^{-a-1-bk^2}G(u_j,q_i), \quad (9.1)$$

where the form of $G(u_j, q_i)$ is to be determined by factorization. When we go to the pole at

$$\alpha(k^2)=a+bk^2=0 \quad (9.2)$$

in (9.1), factorization requires that the residue be expressed in terms of the appropriate form of B_6, Eq. (7.29). Similarly, when we go to the lowest poles on the other internal legs, we want the amplitude to be expressed in terms of the tree diagrams for these configurations. We therefore get the form

$$S(q_1\cdots q_4)=-g^4\int d^4k$$

$$\times\int_0^1 du_1du_2du_3\,du_4u_1^{-a-1-b(k+q_1)^2}$$

$$\times u_2^{-a-1-b(k+q_1+q_2)^2}u_3^{-a-1-b(k-q_4)^2}u_4^{-a-1-bk^2}$$

$$\times[(1-u_1)(1-u_3)]^{-a-1-bs}[(1-u_2)(1-u_4)]^{-a-1-bt}$$

$$\times[(1-u_1u_2)(1-u_2u_3)(1-u_3u_4)(1-u_4u_1)]^{a+b(s+t-m^2)}$$

$$\times[(1-u_1u_2u_3)(1-u_1u_3u_4)]^{b(2m^2-t)}$$

$$\times[(1-u_2u_3u_4)(1-u_1u_2u_4)]^{b(2m^2-s)}H(u_j,q_i), \quad (9.3)$$

where $q_i^2=m^2$, $(q_1+q_2)^2=s$, $(q_1+q_4)^2=t$, and $\alpha(s)=a+bs$ is the unrenormalized linear input Regge trajectory. At this point we still have an undetermined function $H(u_j, q_i)$ which must satisfy the following constraint:

$$H(u_j,q_i)=1, \quad \text{when any} \quad u_j=0. \quad (9.4)$$

This is the result of Kikkawa, Sakita, and Virasoro (1969). They point out that the integral over the loop momentum can be done if a Wick rotation is performed

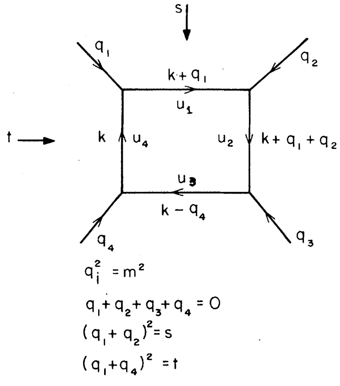

FIG. 9.2. Definition of variables involved in defining the square-graph amplitude of Kikkawa, Sakita, and Virasoro (1969).

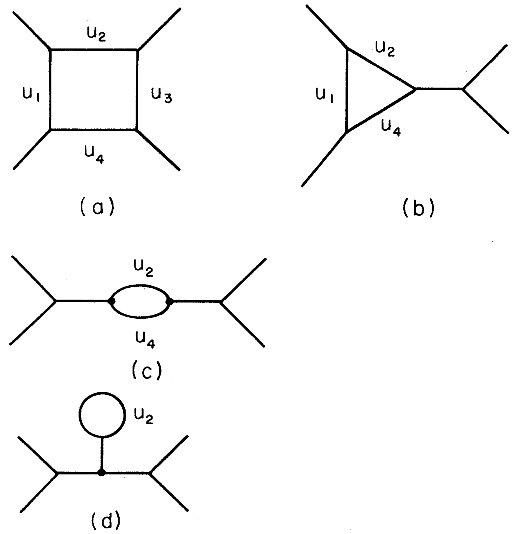

FIG. 9.3. Different corners of the integration region, (u_1, u_2, u_3, u_4), in (9.1) produce different Feynman diagrams. The singularity structure in (a) is produced at (0, 0, 0, 0), a diagram like (b) comes from (0, 0, 1, 0), (c) comes from (1, 0, 1, 1), and (d) from (1, 0, 1, 1). No diagrams are produced at (1, 1, 1, 1) and this is the corner at which the infinite product, (9.9), diverges.

to reach a region where k^2 is negative definite.[48] The asymptotic form of the function is then found to be

$$S\underset{s\to-\infty}{\sim}-g^4\Gamma(-a-bt)(\ln s)(-bs)^{a+bt}$$

$$\times\Bigg[\int_0^1\frac{dx_1dx_3}{\ln^2x_1x_3}\exp\left\{-t\frac{\ln x_1\ln x_3}{\ln(x_1x_3)}\right\}$$

$$\times\{(1-x_1)(1-x_3)\}^{a+bt-1}(1-x_1x_3)^{-2bt}$$

$$\times\left\{\frac{1}{\ln(x_1x_3)}-\frac{x_1}{(1-x_1)^2}-\frac{x_3}{(1-x_3)^2}\right\}^{a+bt}(x_1x_3)^{-a-1}\Bigg]$$

$$(9.5)$$

$$\underset{s\to-\infty}{\sim}-g^4\Gamma(-a-bt)(\ln s)(-bs)^{a+bt}\sum(t), \quad (9.6)$$

which, to second ordering gives the new output Regge trajectory

$$\alpha_{\text{new}}(t)=a+bt+g^2\sum(t). \quad (9.7)$$

It is possible, therefore, that we can maintain crossing symmetry and Regge behavior in a perturbation theory of this sort where internal states include an arbitrary number of internal loops. Polkinghorne (1969) has discussed the interpretation of this renormalized Regge trajectory, which has a nonzero imaginary part and gives poles on the second sheet of the Mandelstam variables.

The reader will note that the integrand of (9.3) can be expanded in a power series in the u_j and that the divergence of this series at the corners of the integration

---

[48] This rotation is performed formally without taking into account any possible contribution from $|k_0|=\infty$.

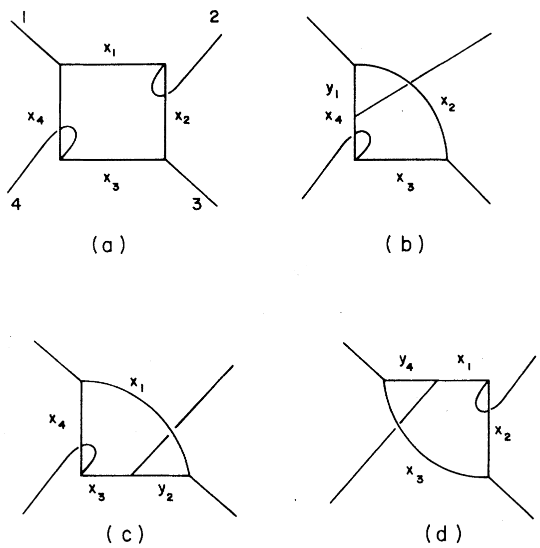

FIG. 9.4. Nonplanar diagrams classified by Kikkawa, Klein, Sakita, and Virasoro (1969).

volume produces the different Feynman graphs indicated by Fig. 9.3. (As we shall see below the corner where all $u_j=1$ contains essential rather than Feynman singularities.)

Our derivation so far depends only on the form of the amplitude B_6 for $J^P=0^+$ bosons and does not take into account the couplings to spinning particles, both on the parent trajectory and on daughter trajectories, which can be projected out of the Veneziano model on the basis of the factorization of Fubini and Veneziano (1969) and of Bardakci and Mandelstam (1969). If we go to a pole at $\alpha(k^2)=n$ in the integrand of (9.3) and require that the residue be consistent with the couplings of the factorized states in Eq. (7.23) and (7.24), we get a further constraint on the form of $H(u_j, q_i)$ in (9.3). This has been done by Bardakci, Halpern, and Shapiro (1969). [See also the note added in proof to Kikkawa, Sakita, and Virasoro (1969).] They show that complete factorization of this type involves replacing each simple factor in (9.3) by an infinite product

$$(1-u_1)^{-a-1-bs}\rightarrow \prod_{n=0}^{\infty} [1-u_1(u_1u_2u_3u_4)^n]^{-a-1-bs}$$

$$\vdots \qquad (9.8a)$$

$$(1-u_2u_3)^{a+b(s+t-m^2)}\rightarrow \prod_{n=0}^{\infty} [1-u_2u_3(u_1u_2u_3u_4)^n]^{a+b(s+t-m^2)}$$

$$\vdots \qquad (9.8b)$$

etc. and including another infinite product which has a form which depends upon the linear dependences or Ward identities among the factorized states in (7.24):

$$\prod_{n=1}^{\infty} [1-(u_1u_2u_3u_4)^n]^{-8m^2-3}. \qquad (9.9)$$

When we put the infinite products into the integrand, we encounter an alarming problem. The infinite products diverge violently at one corner of the integration volume ($u_1=u_2=u_3=u_4=1$). If we arbitrarily remove this piece of the region of integration, the infinite products do not affect the asymptotic behavior, (9.4), of the function as Re $s\rightarrow-\infty$, but we get some exponentially increasing asymptotic form in the Re $s>0$ region which depends on the volume removed. This divergence or exponential increase appears to be due to the large number of daughter states present in the factorized form (7.45) and Mandelstam (1970b) has conjectured that it is necessary to do some sort of renormalization to minimize the importance of these lower trajectories *before* enforcing factorization, in order to get a finite result.[49]

This simple discussion of a square graph illustrates the techniques which can be used to write amplitudes with a single planar loop. The extension to a larger number of external particles, nonequal intercepts, and internal symmetries can be readily constructed. (Again, not without complications however.)

The derivation of the form of the square graph has been recently redone by Amati, Bellac, and Olive (1969) in terms of the operator formalism of Fubini, Gordan, and Veneziano (1969). Since the levels of the harmonic oscillator operators in this formalism provide convenient labels for the factorized internal states in the model, this calculation verifies that the loop is really constructed from a unitary sum. For example, using this method it can be explicitly seen that the linear dependences serve to remove unwanted internal states from the unitarity sum. The calculation reproduces the result of Kikkawa, Virasoro, and Sakita (1969) and of Bardakci, Halpern, and Shapiro (1969).

B. Twisted Loops, Nonplanar Diagrams, and Regge Cuts

Once we start considering internal loops, we are led to Feynman diagrams which are nonplanar. Experience with sums of Feynman diagrams [see C. Risk* (1968) for a review of the asymptotic behavior of sums of Feynman diagrams] suggests that functions with a nonplanar singularity structure will have a more complicated asymptotic behavior. They may, for example, have cuts rather than poles in the J-plane.

In the Veneziano model, one way of looking at this problem involves the twisting operator (Amati, Bellac, and Olive, 1969; Caneschi, Schwimmer, and Veneziano, 1969) mentioned in Sec. VII. Recall that $B_N(p_1, \cdots, p_N)$

[49] Whether the absorptive part of the box diagram increases exponentially as $s\rightarrow+\infty$ is not known. Since all known renormalization procedures do not affect absorptive parts, such an event would have calamitous implications for the future of this theory. As yet no one has discovered a way of properly deforming the contour of the loop integral in (9.3) so as to perform the calculation directly. An indirect computation may be possible by examining the total width of a state on the parent trajectory as a function of its mass.

depends on the ordering of the external particles. Different orderings are related by twists of the internal lines of the tree diagram. From the schematic indication of how a loop is "sewn together" from the tree diagram given by Fig. 9.1 we can see that, in general, the presence of a twist can change the result. Loops containing twisted lines are commonly referred to as twisted loops.

Kikkawa, Klein, Sakita, and Virasoro (1969) have classified the various nonplanar diagrams related by duality (see Fig. 9.4) to be either orientable or nonorientable according to whether the diagram contains an even or an odd number of twisted lines. The terms "orientable" and "nonorientable" refer to the topological structure of the dual diagrams. They have shown that the reasoning which led to (9.3) can be repeated to formulate a recipe for constructing functions with cut singularities.

Kikkawa (1969) has taken a simple example of a function with a nonplanar loop without the complication of the infinite products and has shown that it possesses an asymptotic behavior which corresponds to a Regge cut. This result makes plausible a connection between this model and sums of Feynman diagrams. It also indicates that if a convergent perturbation series based on the Veneziano model could be formulated, it would probably contain Regge cuts, which seem to be desirable from a phenomenological viewpoint (see the review of Jackson,* 1970).

Thorn and Kaku (1970) have used the harmonic oscillator formalism of Fubini, Gordan, and Veneziano (1969) and the twisting operator of Caneschi, Schwimmer, and Veneziano (1969) to perform the unitary sums present in a diagram with one nonplanar loop. Their result agrees with Kikkawa, Klein, Sakita, and Virasoro (1969).

C. Ghosts, Factorization, and Divergences

The divergences present in the square graph of this model due to the infinite products make the interpretation of the results very difficult. Olesen (1970b) has shown that, by adding more trajectories of nonleading intercept, it is possible to remove the divergences in the square graph. The divergence which appears in the simple model with a universal trajectory is due to two things: (a) the large number of states and (b) the bad behavior of the vertex operator. Introducing the new trajectories *à la* Olesen increases the number of states, but also results in a much better behaved vertex function.

The infinite products present in nonplanar loop diagrams diverge at various places within the integration volume. It is not possible to remove one specific chunk of the integration volume which removes divergences from all types of diagrams.

To interpret the results of this model we need a renormalization scheme apparently several orders of magnitude more complicated than renormalization in quantum electrodynamics. Much effort is being expended to solve this problem, and various calculations are being made on the assumption that a renormalization scheme will be found. In particular, an effort is being made to construct functions containing more than one internal loop. The possibilities for progressive complication seem endless. What is lacking is a substantive clue that this approach has a reasonable chance of realistically describing hadron physics.

Because of the presence of ghosts, associated with the indefinite metric rising from the orbital factor structure of Sec. VII.C, it is even quite plausible that the theory can never be made unitary and analytic at the same time.

The prototype theory of this kind is that discussed by Lee and Wick (1968) and Lee (1969). The point is that, in each order of "perturbation" theory, the ghosts in such a model are likely to lead to negative cross sections. Whether or not this is actually the case here needs to be checked. We suspect that it is, and that the problem needs to be avoided via the methods of Lee. If this problem is present, then the probability that the whole method is at all relevant seems infinitesimal.[50]

See also the work of Green (1970), Susskind (1969b), and Thorn (1969).

X. PHENOMENOLOGY AND RELATED MATTERS

In this section we will discuss various aspects of the question: Is there any experimental evidence which lends support to the idea that the narrow resonance model is an approximate description of reality? In particular, are there any pieces of the model which could reasonably be used instead of, or in addition to, the already available multiperipheral model (Chew, Goldberger, and Low, 1968), the strip model (Collins and Johnson, 1969), or the absorptive Regge model (Arnold, 1968)? See Jackson* (1970) for a review of all these models and their applications.

With certain qualifications, our answer to these questions is *no*. In this section, we will explain this conclusion by discussing specific examples.

A. Existence of Subsidiary Trajectories

Suppose we do not worry about factorization and instead concentrate on a specific, isolated interaction. As discussed in Sec. III, the narrow resonance model predicts an infinite set of resonance towers; each tower being a set of mass degenerate states with spins running from zero up to a maximum value, $\alpha(M^2)$, where M^2 is the tower $(\text{mass})^2$ and $\alpha(x)$ is the leading Regge trajectory. The model for the four-point function predicts the elastic width of each state in the tower, although, in view of the fact that unitarity is violated

[50] We are indebted to R. F. Dashen for pointing out to us the possible connection between the narrow resonance bootstrap and the model of Lee and Wick (1969). That diseases of this kind can occur in every order has also been noticed by D. Amati.

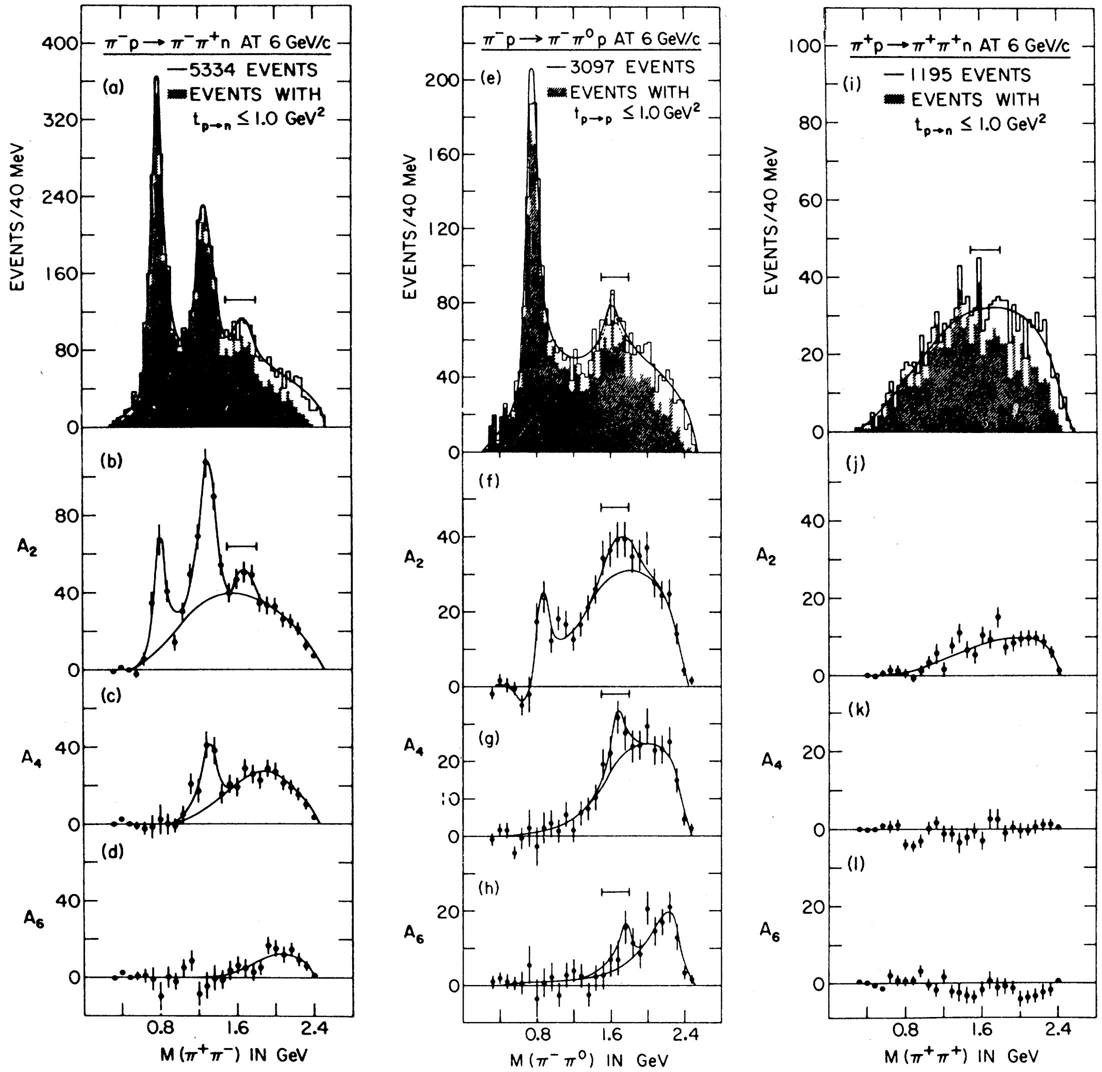

FIG. 10.1. Pion–pion mass distributions and Legendre polynomial coefficients for $\pi N \rightarrow \pi\pi N$ at 6 GeV/c from Crennel *et al.* (1968). The columns are, from left to right, for $\pi^- p \rightarrow \pi^+ \pi^- n$, $\pi^- p \rightarrow \pi^- \pi^0 p$, $\pi^+ p \rightarrow \pi^+ \pi^+ n$.

in the model, we are uncertain how seriously we may interpret these elastic width predictions.

Consider again the $\pi\pi$ narrow resonance amplitude discussed in Sec. III. The model contains a 0^+ resonance (the ϵ) degenerate with the ρ, with a partial width given by

$$\Gamma_\epsilon / \Gamma_\rho \cong 9/2. \qquad (10.1)$$

Since this resonance would be quite broad, its presence must be inferred from the gross behavior of the $I=0$ s-wave $\pi\pi$ phase shift, $\delta_0{}^0(s)$, from threshold up to 1 GeV. The ϵ resonance has been invoked in the past for various reasons and its status is still controversial.[51] Without going into details, we conclude that the existence of a resonance with the predicted properties does not contradict available experimental evidence, nor is such an object strongly required to fit existing data.

At the mass of the f^0, the model predicts that a $J^P = 1^-$ particle (the ρ') exists with

$$\Gamma_{\rho' \rightarrow \pi\pi} / \Gamma_{\rho \rightarrow \pi\pi} \cong 1 \qquad (10.2)$$

and that if a 0^+ state (the ϵ') exists, it does not couple to the $\pi\pi$ system. The prediction (10.2) seems in disagreement with experiment. In Fig. 10.1 we show the data of Crennel *et al.* (1968) for the process $\pi^- p \rightarrow \pi^+ \pi^- n$ and for $\pi^- p \rightarrow \pi^0 \pi^- p$. There it will be noted that the ρ^0, f^0, and g^0 appear in the $\pi^+\pi^-$ invariant mass plot, but that there is no signal at all in the $\pi^-\pi^0$ invariant mass distribution at the f mass, though the ρ^- and g^- show

[51] See the many and conflicting experimental and theoretical excursions in "Proceedings of the ANL Conference on $\pi\pi$ and πK Interactions," May 1969.

up nicely. This does not directly test (10.2) because the data measure $d\sigma(\pi^- N \rightarrow \rho' N)\Gamma(\rho' \rightarrow \pi\pi)/\Gamma(\rho'$ total) and the production mechanism may, for some reason, be small. Jackson and Quigg (1968) have suggested a way of estimating the production. They point out that the absorptive one pion exchange (OPEA) model has proved reliable for computing ρ production by pions. Treating the ρ' as a heavy ρ, the OPEA calculation should give a reasonable estimate of the relative production cross sections for ρ' and ρ. The ratio of ρ' to ρ events in the data is not more than 1/10. Combining this with the OPEA estimate and assuming the ρ' is mostly elastic, we have the fairly reliable upper limit

$$\Gamma_{\rho' \rightarrow \pi\pi}/\Gamma_{\rho \rightarrow \pi\pi} < 0.13, \qquad (10.3)$$

nearly an order of magnitude away from the prediction (10.2). Corroberation of the limit (10.3) can be found in the reaction $\gamma N \rightarrow \pi\pi N$ (McClellan *et al.*, 1969).[52]

Since the existence of the ρ' has been predicted by the quark model (Harari,* 1968) and has been invoked to fit electromagnetic form factors (Balachandran, Freund, and Schumacher, 1964; Wilson,* 1966; Cordes and O'Donnell, 1969) and charge exchange polarization (Barger and Phillips, 1968), the absence of this resonance embarrasses others besides the proponents of the Veneziano model. But this is beside the point. A narrow resonance model is certainly no better than the resonance spectrum it predicts and the use of this model for phenomenology will continue to be suspect unless a ρ' resonance is found.[53] One escape would be to assume that the ρ' is very inelastic. If the ρ' had a total width of order 1 GeV or if there were an accumulation of secondary effects, there would be no conflict between the production data and (10.2).

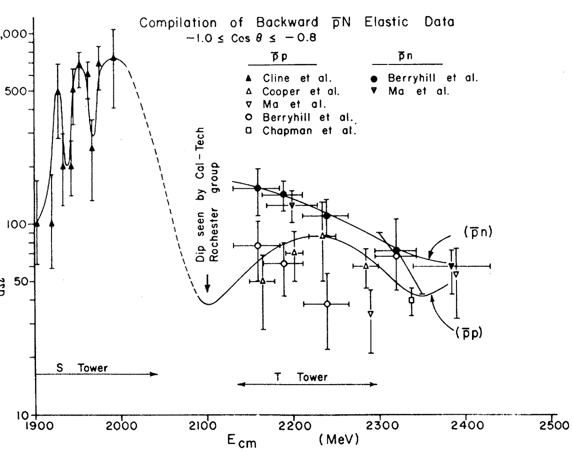

FIG. 10.2. A compilation of all available data on backward hemisphere $\bar{p}p$ elastic scattering data as a function of the center-of-mass energy of the $\bar{N}N$ system, taken from Barger and Cline (1969b). The data in the S region are from Cline, English, Reader, Terrell, and Twitty (1968). The data in the T region are from Berryhill (private communication), Cooper *et al.* (1968), Ma, Parker, Smith, Sprafka, Abolins, and Rittenberg (1968), and Lys, *et al.* (1968) [denoted as Chapman *et al.* on the figure]. An eyeball curve is drawn through the data. A recent counter experiment covering the mass region of 2000–2400 has observed a sharp dip near 2100 (Barish, 1969; Tollestrup and Lobkowicz, private communication). This dip is schematically included in the curve drawn through the $\bar{N}N$ data to indicate the possible separation of the S and T "tower" regions (compare Fig. 2.3). According to Barger and Cline, the narrowness of the fine structure observed in the S "tower" may represent narrow resonance states, but might also come from broader resonant states which are cut off on the low side by centrifugal barrier effects in the $\bar{N}N$ system.

The existence of approximately degenerate meson towers is strongly dependent on the validity of semilocal duality, and in general results in the absence of backward peaks in elastic scattering processes having an exotic u channel. (A somewhat more optimistic view than taken here of the experimental situation can be found in Barger and Cline (1969b), who discuss $\pi^+\pi^-$, π^+K^-, K^+K^-, and $\bar{N}N$ elastic scattering.)

For the meson–meson processes considered by Barger and Cline, there is some doubt, which we share, that the data actually exist.[51] The use of $\bar{N}N$ elastic scattering, on the other hand, does not suffer from this ambiguity, and Barger and Cline propose several methods for detecting meson towers in this reaction. Their compilation of the $\bar{p}p$ and $\bar{p}n$ data for $d\sigma/d\Omega$ up to a center-of-mass energy of 2.5 GeV is shown in Fig. 10.2. The evidence for the tower structure is inconclusive, but the approach is interesting and deserves further investigation.

As we discovered in previous sections, an attempt to construct narrow resonance amplitudes for processes with more complicated crossing structure than that of $\pi\pi \rightarrow \pi\pi$ leads to trajectories with negative widths. Also, in processes with nontrivial helicity crossing matrices there is no compelling reason to restrict attention to simple one-term formulas, so there is no unique daughter structure to discuss.[54] One thing we can say is that a large number of daughter states must exist if we are to maintain the concepts of resonance dominance of absorptive parts and Regge behavior which first led us to investigate narrow resonance models. A possible rationalization for the failure to find such daughters is the interpretation that lower daughters in narrow resonance models actually represent background in the physical amplitudes (Bardakci, 1969). This belief is behind the following statement frequently found in the literature: "The model can only be believed for parents and first daughters" (Lovelace, 1969a). This interpretation of the predictions of narrow resonance models is in striking contradiction with the philosophy of Fubini and Veneziano (1969), Bardakci and Halpern (1969), and Bardakci and Mandelstam (1969) discussed in Secs. VII and VIII.

---

[52] A discussion of the experimental situation vis-à-vis heavy vector mesons can be found in Diebold* (1969), Sec. II.4.

[53] A contrary opinion may be found in Harari* (1969).

[54] Evidently, a simple narrow resonance parameterization of a process with nontrivial external spins is precluded, if one simultaneously tries to eliminate exotic resonances. One of us (J.Y.) would like to thank Professor Lorella Jones for a helpful private communication regarding this problem.

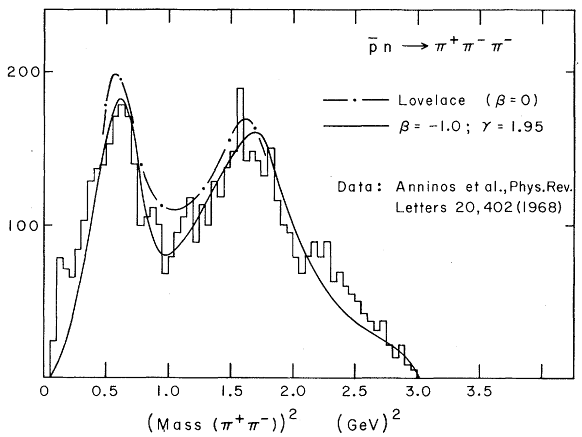

FIG. 10.3. Invariant mass distribution for $\pi^+\pi^-$ from $\bar{p}n \to \pi^+\pi^-\pi^-$. Data taken from Anninos *et al.* (1968). Theoretical curves are those of Lovelace (1969b) and Berger (1969a).

B. The Process $N\bar{N} \equiv 3\pi$

The process $\bar{p}n \to \pi^-\pi^-\pi^+$ for stopping antiprotons has been compared with the Veneziano model by Lovelace (1968), Berger (1969b), and Altarelli and Rubinstein (1969).

Lovelace suggested the use of a two-term formula

$$\beta \frac{\Gamma[1-\alpha(s_1)]\Gamma[1-\alpha(s_2)]}{\Gamma[1-\alpha(s_1)-\alpha(s_2)]} + \gamma \frac{\Gamma[1-\alpha(s_1)]\Gamma[1-\alpha(s_2)]}{\Gamma[2-\alpha(s_1)-\alpha(s_2)]} \qquad (10.4)$$

with a phenomenological Regge trajectory

$$\alpha(x) = 0.483 + 0.885x + ic(x-4m_\pi^2)^{1/2}\theta(x-4m_\pi^2) \qquad (10.5)$$

and with $(p = p_1 + p_2 + p_3)$,

$$s_1 = (p+p_1)^2, \quad \text{and} \quad s_2 = (p_1+p_2)^2, \qquad (10.6)$$

where p_1 and p_2 are the 4-momenta of the two π^- mesons. Since the trajectory (10.5) has an imaginary part, the poles in (10.4) are no longer on the real axis. Also, their residues are no longer polynomials in the crossed-channel invariant, but the ancestor problems associated with this property are numerically not too serious for this particular application.

The plausibility of the form (10.4) arises from the experimental fact that $\bar{p}n$ annihilation at rest proceeds mainly through the singlet state so the initial system acts like a heavy pion. Then (10.4) can be considered as arising from some sort of mass extrapolation of one leg of the $\pi\pi \to \pi\pi$ system.

Lovelace set $\beta = 0$ and attempted to find a one-parameter [c in (10.5)] fit to the Dalitz plot and to the $\pi^+\pi^-$ and $\pi^-\pi^-$ mass distributions. Berger, and Altarelli and Rubinstein pointed out independently that this fit did not match the angular distribution in the ρ and f regions. Berger took β and γ in (10.4) to be free parameters and found a best fit which is compared in Figs. 10.3–10.6 to a fit using Lovelace's parameters. Not surprisingly, Berger's fit is somewhat of an improvement. Altarelli and Rubinstein take three more terms than in (10.4) and, having nine parameters, do slightly better.

The claim that such fits provide evidence for the Veneziano model is debatable. As Berger points out, it is not clear how the details of the model have entered beyond the fact that the $\pi\pi$ system contains a ρ, an f, and a large s-wave phase shift. However, the data of Anninos *et al.* (1968) do show a diminution of events on the Dalitz plot near $s_1 = s_2 \cong 1.1(\text{GeV})^2$ and indications of possible zeros at $s_1 \cong 0$, $s_2 \cong 2.1$ and $s_1 \cong 2.1$, $s_2 \cong 0$, so this point remains an open question.

Altarelli and Rubenstein (1969) and Jengo and Remiddi (1969a) have also considered the annihilation $\bar{p}p \to 3\pi$ where the Dalitz plot is not so well known. Using Bizzarri's (1968) estimate of the conversion factor $\bar{p}p \to \text{all}/\bar{p}n \to \text{all}$, Altarelli and Rubenstein reach rough agreement between the experimental and theoretical values of

$$\bar{p}n \text{ vs } \bar{p}p \underset{(I=1)}{\longrightarrow} \pi^+\pi^-\pi^0.$$

Jengo and Remiddi also discuss the total rates using Lovelace's original form, $\beta = 0$ in (10.4). They compute

$$R = \Gamma(\bar{p}n \to \pi^+\pi^-\pi^-)/\Gamma[\bar{p}p(^1s_0) \to \pi^+\pi^-\pi^0] = 0.17, \qquad (10.7)$$

which conflicts with the theoretical calculation of Altarelli and Rubinstein and also with their phenomenological estimate:

$$R = 1.6(+1.1, -0.8).$$

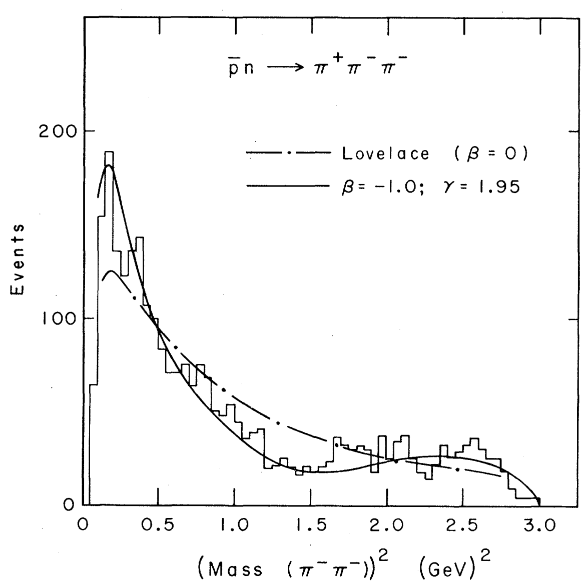

FIG. 10.4. Same as Fig. 10.3 for invariant mass distribution of $\pi^-\pi^-$.

Note added in proof: Recent further experimental work on $\bar{p}n$ annihilation in flight reveals what are evidently more zeros scattered across the (expanded) Dalitz plot. Odorico (1971b) has made the interesting observation that the totality of zeros evidently does not correspond to the usual $\Gamma\Gamma/\Gamma$ form but rather a better fit is obtained by using $\Gamma\Gamma/\Gamma$ times

$$\sin[\tfrac{1}{2}\pi(x-y)]/\sin[\tfrac{1}{2}\pi(x+y)].$$

Since this new function is odd rather than even under crossing, we are facing a situation that makes no sense in the language above.

C. K-Matrix Unitarization Procedure

From the arguments in Sec. X.A, the $\pi\pi$ narrow resonance model cannot be believed in the region of the f mass, but suppose it can be believed below the f. That is, suppose we believe in the ρ and ϵ resonances predicted by the model. Can the model be "improved", beyond the obvious use of Breit–Wigner line shapes, by a simple K-matrix unitarization in order to give a believable set of phase shifts?

Lovelace (1969a) has suggested that this procedure will give consistent phase shifts. (See further discussion above in Sec. VI.B.) The K-matrix approach has been applied by Wagnet (1969b) to the process $\pi N\rightarrow \pi\pi N$ and by Roberts and Wagner (1969b) to K_{14} decay. Lovelace himself (1969a) has applied the method to a coupled $\pi\pi$-$K\bar{K}$ system and compared the results with other semiexperimental analyses.

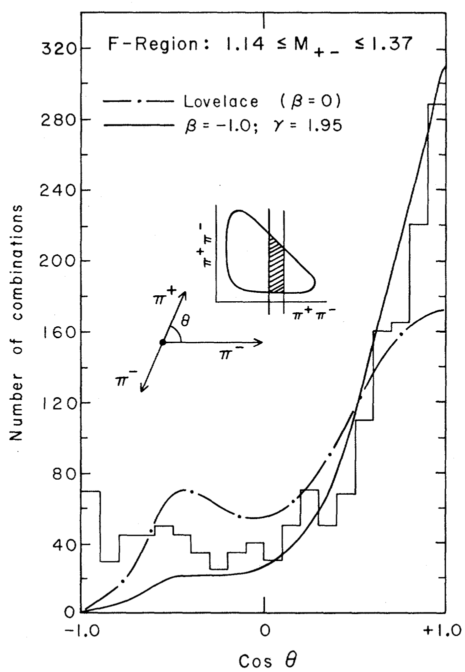

FIG. 10.5. Distribution in the Dalitz angle, defined by the inset, for events in the f^0 mass region in $\bar{p}n\rightarrow\pi^+\pi^-\pi^-$. Experimental and theoretical curves as in Fig. 10.3.

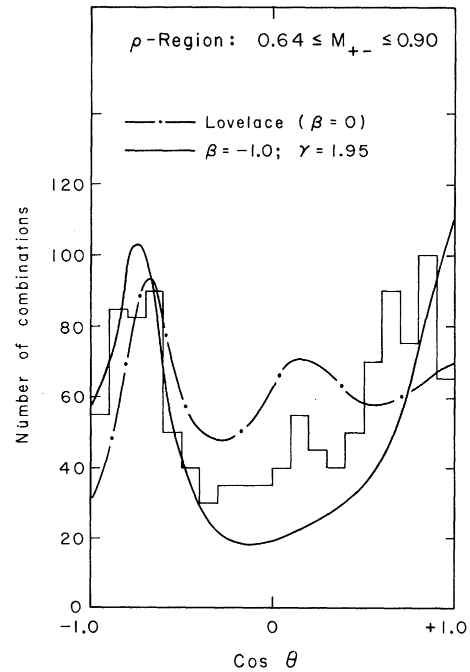

FIG. 10.6. Same as Fig. 10.5 for ρ mass region.

Recall that the K-matrix formalism essentially enforces elastic unitarity so that the low-lying resonances predicted by the model are given a total width approximately equal to their elastic widths. The procedure destroys the crossing symmetry of the amplitude, so the phase shifts cannot be completely consistent.[55] To see this, we assume that the $\pi\pi$ $I=1$, t-channel amplitude satisfies an unsubtracted dispersion relation, so that we have the on-shell form of Adler's $\pi\pi$ sum rule (Adler, 1965a):

$$\tfrac{1}{6}(2a_0-5a_2)\equiv L=\frac{m_\pi^2}{6\pi}\int_{2m_\pi^2}^{\infty}\frac{d\nu}{(\nu^2-4m_\pi^2)}$$
$$\times[2A_0(\nu,0)-5A_2(\nu,0)+3A_1(\nu,0)]. \quad (10.8)$$

This relation may be more recognizable to readers in the form

$$L=\frac{m_\pi}{8\pi^2}\int_{2m_\pi^2}^{\infty}\frac{d\nu}{(\nu^2-4m_\pi^2)^{1/2}}[\sigma^{+-}(\nu)-\sigma^{++}(\nu)], \quad (10.9)$$

where σ^{ab} is the total cross section for $\pi^a+\pi^b$. Now, in the Veneziano $\pi\pi$ formula, Eq. (3.13), with zero mass pions and $\alpha_\rho(0)=\frac{1}{2}$, $\alpha_\rho'(0)=1$, the sum rule can be written as

$$L=\pi L(\rho), \quad (10.10)$$

where $L(\rho)$ is the ρ contribution to the right-hand side of (10.8). If we take $\alpha_\rho(0)=0.48$ and use the physical pion mass, we get

$$L=1.05\pi L(\rho). \quad (10.11)$$

[55] The same objection evidently applies to the Padé approximate method of Basdevant and Lee (1969).

Table 10.1. $\pi\pi$ Scattering lengths (theoretical).

Source	$L=(1/6)(2a_0-5a_2)$	a_0/a_2	Remarks
Adler[a]	$0.10\pm0.01/m_\pi$[h]	⋯	$SU(2)\otimes SU(2)$ and Goldberger–Treiman relation
Weinberg[b]	Same	$-7/2$	Broken $SU(2)\otimes SU(2)$ via $(\frac{1}{2},\frac{1}{2})$ representation
Tryon[c]	$0.11\pm0.01/m_\pi$[i]	$-7/2$	Unitary, crossing symmetric numerical procedure
Morgan and Shaw[d]	$0.10\pm0.01/m_\pi$[j]	-3.2 ± 1.0[g]	Numerical unitarization of fixed t dispersion relations, input $\Gamma_\rho=120$ MeV, $\delta_0^2(m_\rho)=-20°$, $m_\rho=764$ MeV, δ_0^0 in ρ region
0th Order Veneziano[e]	$0.11\pm0.02/m_\pi$	⋯	Error from uncertainty in rho width; a_0/a_2 undetermined
Lovelace I[f]	$0.15/m_\pi$	-4.5	Threshold Phase Shift from "K-Matrix" procedure, coupled $\pi\pi$–KK channels
Lovelace II (Tryon)[e]	$0.12/m_\pi$ $0.13/m_\pi$	⋯	Numerical integration of Lovelace phase shifts using $\pi\pi$ sum rule; upper value coupled channel, lower uncoupled
Lovelace III (Morgan and Shaw)[d]	$0.11/m_\pi$	-14.5	Numerical unitarization of fixed t-dispersion relations using Lovelace δ_0^0 in ρ region and $\delta_0^2(m_\rho)$

[a] Adler (1965a).
[b] Weinberg (1966).
[c] Tryon (1969a).
[d] Morgan and Shaw (1970).
[e] Tryon (1969a).
[f] Lovelace (1969a).
[g] Gutay *et al.* (1969); Cline *et al.* (1969).
[h] Error estimate from assuming the error in the $\pi\pi$ extrapolation $\leq$ the error in the πN (g_A) sum rule.
[i] Error arising from inherent uncertainties in numerical procedure; see text of Footnote c above.
[j] Error arising from uncertainties in input from semiphenomenological analyses of Footnote g above.

Finally, normalizing by taking the ρ width to be 112 MeV, we get $L=0.108m_\pi^{-1}$. This is the naive value one gets from the narrow resonance model. It essentially agrees with the current algebra value of Adler (1965a). On the other hand, Lovelace's K-matrix form gives $L=0.15m_\pi^{-1}$ (Lovelace, 1969a). This discrepancy between Lovelace's result and the current algebra result casts suspicion on Lovelace's low-energy phase shifts since the corrections to the current algebra value arising from mass extrapolations are expected to be equal to or less than the error in the g_A sum rule for πN scattering, which is about a 10% correction.[55]

The sum rule (10.8) provides a rather delicate test of crossing symmetry. The crossing properties of the first few partial waves can be improved to a certain extent by an iteration procedure in such a way to obtain a modified set of ρ and ϵ parameters. This has been done in several ways for the region below 1 GeV in $\pi\pi$ scattering by Tryon (1969a) and by Morgan and Shaw (1970). Tryon's model is exactly crossing symmetric, is approximately unitary in the energy region below 1 Gev, and converges to the narrow resonance amplitude (3.13) in the asymptotic region. The K matrix phase shifts of Lovelace can be inserted into (10.8) and L can be obtained. Tryon (1969b) has done this calculation and obtained values of $L=0.12m_\pi^{-1}$ or $L=0.13m_\pi^{-1}$, depending on whether or not a coupled $K\bar{K}$ channel is included. Morgan and Shaw have also used Lovelace's phase shifts in their crossing symmetric procedure and have obtained $L=0.11m_\pi^{-1}$. Independently of the coupled channel problem, these calculations indicate that crossing symmetry is important, even at low energies, and dramatize the danger of using phenomenological forms which violate it. The situation is summarized in Table 10.1.[56]

There have been attempts to use the K-matrix procedure to go even further and fit off-shell behavior. For $\pi N\rightarrow\pi\pi N$ this has been done by Wagner (1969b), and Roberts and Wagner (1969a). Wagner's fit to $\pi N\rightarrow\pi\pi N$ uses one-pion exchange and depends on a modification of the K-matrix procedure, the off-shell partial-wave amplitude being given by

$$f_{l\,\text{off}}^{(I)}=ga_{l\,\text{off}}^{I}(s,q)/[1+\rho_l^I ga_l^I(\text{on})], \quad (10.12)$$

where by $a_{l\,\text{off}}^I(s,q^2)$ we mean the amplitude for

[56] Though the discrepancies listed in Table 10.1 are not large, they reveal that the K-matrix method does not add to our understanding of the low-energy $\pi\pi$ interaction. Quite to the contrary, in comparison with the zeroth order Veneziano term and its resonance parameters, it seems to detract.

TABLE 10.2. The partial wave analysis of the resonance tower at the position of the $F_{17}\Sigma(2030)$ for two narrow resonance solutions for KN and $\bar{K}N$ scattering of Berger and Fox (1969). The kinematic factors have been evaluated at the pole positions predicted by the theoretical trajectories. We list $\Gamma_{\bar{K}N}$(MeV). The experimental width of the $\Sigma(2030)$ is presently given as 80–170 MeV, with a $\bar{K}N$ branching fraction of 10%–27%. Solution (A) was constructed by using the arguments of Inami (1969) and solution (B) is a somewhat more complicated construction of Berger and Fox (1969). Details of the fitting procedure can be found in Sec. III of Berger and Fox. Both high-energy and resonance region data have been used in forming (A) and (B). As discussed in the text, the $5/2^+$ state here is related by $SU(3)$ to the $F_{35}\Delta(1890)$.

Solution (A)				Solution (B)			
J^P	Γ	J^P	Γ	J^P	Γ	J^P	Γ
$1/2^+$	15.7	$1/2^-$	8.9	$1/2^+$	21.0	$1/2^-$	52.5
$3/2^-$	2.7	$3/2^+$	8.3	$3/2^-$	−18.3	$3/2^+$	−5.9
$5/2^+$	−0.3	$5/2^-$	9.0	$5/2^+$	5.7	$5/2^-$	5.0
$7/2^-$	0.2	$7/2^+$	8.9	$7/2^-$	2.2	$7/2^+$	29.9

$\pi\pi\rightarrow\pi\pi(q^2)$, one of the external legs having $q^2\neq m_\pi^2$. Wagner assumes that the constant, g, in (3.13) is replaced by[57]

$$g \exp[-\gamma(m_\pi^2-q^2)] \qquad (10.13)$$

and uses, to compute partial wave amplitudes,

$$t=-\tfrac{1}{2}(s-3m_\pi^2-q^2-4\,|\,\mathbf{q}_{\text{off}}\,||\,\mathbf{q}_{\text{on}}\,|\cos\theta). \quad (10.14)$$

Equations (10.12) and (10.13) amount to a rather arbitrary prescription and in fact, in order to fit the data, Wagner is forced to introduce a subtraction constant into ρ_l^I, for $l=I=0$. Because of the arbitrary nature of the assumptions involved, we do not believe that Wagner's fits embody a test of the underlying model or even of the K-matrix procedure. The K-matrix procedure has also been applied to K_{l4} decay by Roberts and Wagner (1969b). We have similar objections to this calculation, and will not discuss K_{l4} decay further here.

D. Meson Baryon Scattering

In this section we discuss the phenomenology of πN and KN scattering from the narrow resonance point of view. As we have pointed out at length above, narrow resonance models are extreme forms of pure Regge pole models, in the sense that in physical regions high-energy behavior is governed by moving powers, with residues of definite form. We can therefore expect that any difficulties already present in classical Regge phenomenology will continue if we try to use narrow resonance forms to fit data. As we shall see below, this is precisely what happens.

The relevant work is by Amann (1969), Berger and Fox (1969), Igi (1969), Igi and Storrow (1969), Inami (1969), Lovelace (1969b), Pretzl and Igi (1969), Virasoro (1969c), and Fenster and Wali (1970). Generally we will follow the arguments of Berger and Fox.

We will examine the following questions:

(a) How are the narrow poles to be smoothed over?

(b) To what extent is the atonous duality property and the satellite structure reflected in the data?

(c) Can parity doublets be eliminated, so that the narrow resonance spectrum is reasonably related to reality?

(d) Regge residues in this model, as we have discussed above, take the form

$$\beta_n(\alpha)=(q_tq_t')^{\alpha-n}[P_n(\alpha)/\Gamma(\alpha-n+\tfrac{3}{2})], \quad (10.15)$$

with the usual threshold factor and Mandelstam zeros multiplied by a polynomial in α. How does this agree with the data?

With respect to (a), a completely satisfactory way of smoothing over the narrow poles does not yet exist. In the literature, this question is usually avoided by choosing a complex trajectory, since none of the "unitarizations" mentioned in Sec. VI has been useful in making detailed fits.[58]

As for question (b), the satellite structure has proved a great roadblock to taking all details of the model seriously, since, as emphasized above, there is not even one known resonance which can unambiguously be identified as lying on a satellite trajectory.[53] In Table 10.2, we show a computation by Berger and Fox of the widths of the tower of states degenerate with the $\Sigma(2030, 7/2+)$, arising from two different narrow resonance solutions for KN and $\bar{K}N$ scattering, which we will discuss below. Berger and Fox tried to identify some of the states in this tower with the $SU(3)$ partners of known resonances [e.g., the $5/2+$ with $\Delta(1890, 5/2+)$, etc.], which have been classified in the quark model (Harari,* 1968; Morpurgo,* 1968). For solution

[57] This form factor effectively Reggeizes the pion. Compare the form factors of the absorption model. (Ferrari and Selleri, 1962; Jackson and Pilkuhn, 1964). We will discuss the $q^2=t$ distribution further in X.E. It should be noted that there is no evidence in nature for Reggeized pions (Berger, 1969a).

[58] Alternatively, one can ignore the fact that there are real poles in the physical regions and use the narrow resonance pole residues as parameters to be fit with empirical elastic widths. This was done by Berger and Fox.

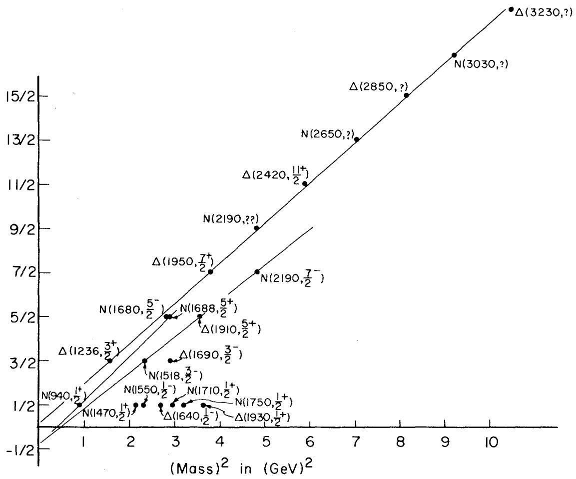

Fig. 10.7. Fanciful Chew–Frautschi plot of known N, Δ resonances. Data from Rosenfeld *et al.** (1969).

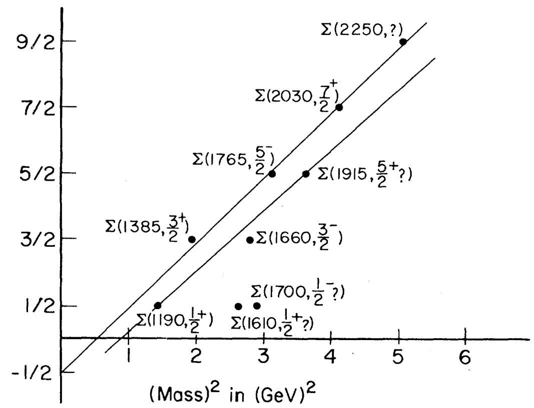

Fig. 10.9. Same as Fig. 10.7 for Σ resonances.

A, in Table 10.2, the width of the $5/2+$ state is clearly unreasonable, either if one uses SU_3 to relate it to the observed $F_{35}(1890)$, or puts it in the quark model (**56, 2**). In the other solution, this width has become positive at the expense of making the s wave huge and creating p and d wave ghosts. As discussed in X.A above, probably one needs to rationalize away satellites if one insists on using the model phenomenologically, either by saying they really represent background or by insisting they arise from local duality and that duality is badly violated in this energy region.

The question of parity doublets, (c), is again a difficult one. In Figs. 10.7–10.10 we show a fanciful version of Chew–Frautschi plots for most of the proposed baryon resonances. Parity doubling is not much in evidence, while generally, as has been known for a long time, (Gribov, 1963) resonance models for processes with external spins generally have all trajectories parity doubled, as discussed above in Sec. V.

As pointed out by Berger and Fox, one can always add subsidiary terms to cancel parity doublets along the leading trajectory: however, if one attempts this for the lower trajectories, the Regge behavior of the amplitude will be lost.

As for question (d), there is an important difference between the residue functions used for fits, for example, by Barger and Phillips (1969) and Barger (1969a), and those found in the Veneziano model. In the former, exponential dependence is either introduced explicitly or implicitly by adjusting the "scale factor," s_0, while in the narrow resonance model the "scale factor" is constrained to be b^{-1}, where b is the universal trajectory slope and residues are determined up to a polynomial, as in (10.15).

Since the Regge residues in the model are no longer arbitrary, we can relate asymptotic behavior along the fixed u direction to the baryon trajectories as shown in Fig. 10.11. Berger and Fox found that the Veneziano parameterization does not provide an accurate extrapolation for the Δ trajectory. They find the best fit

$$\alpha_\Delta(u)=0.09+0.9u, \tag{10.16}$$

$$\gamma_\Delta(u)=[\alpha_\Delta(u)-\tfrac{1}{2}]\{35.2+56.0u$$
$$+[(u)^{1/2}-M_N][29.4+35.8u]\} \tag{10.17}$$

for the Δ extrapolation. This is shown in Fig. 10.12, where it can be seen that the coefficients of u and $u^{3/2}$ above, indicative of large subsidiary terms, cause γ_Δ

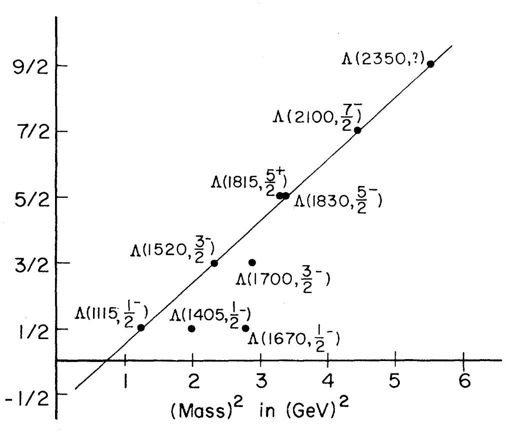

Fig. 10.8. Same as Fig. 10.7 for Λ resonances.

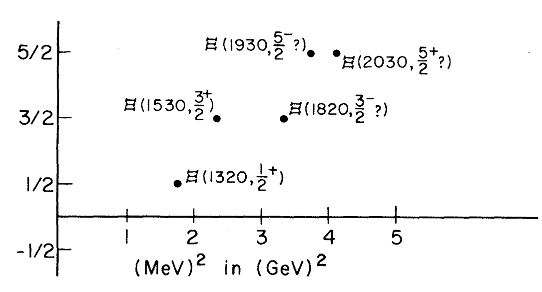

Fig. 10.10. Same as Fig. 10.7 for Ξ resonances.

to grow too rapidly at large $|u|$.[59] Further, though they found that residues of states on the N_α, $\Sigma_\beta-\Sigma\gamma$, and $\Lambda_\alpha-\Lambda\gamma$ trajectories were related well by the model to backward data, this was at the expense of including rather large contributions from nonleading terms. For KN and $\bar{K}N$ scattering, though Berger and Fox allow themselves a great variety of terms, of course accompanied by a large number of free parameters, they are unable to overcome discrepancies in over-all amplitude magnitude versus the K^+n CEX data, and in the t distribution for K^+p backward scattering. They conclude these troubles are due to effects lying outside the basic model.

Finally, there are the classical difficulties alluded to above, associated with high-energy elastic diffraction, dips, and the so-called "crossover zero." We will

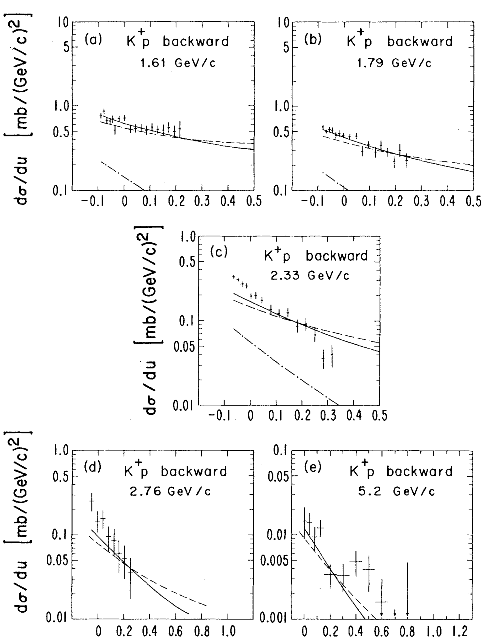

FIG. 10.11. K^+p backward scattering data as taken from Carroll *et al.* (1968), Abrams *et al.* (1968), Cline *et al.* (1967), Banaigs *et al.* (1969), and Baker *et al.* (1968). The dashed curve is solution (A) and the solid curve is solution (B), both from Berger and Fox (1969). The dot-dashed curve is computed using the expressions for the A and B amplitudes of solution (B), keeping the leading asymptotic term only. As is apparent, in this approximation with only the leading Regge trajectory, one underestimates the empirical $d\sigma/du$ badly at the lower energies.

[59] The various workers in this field do not seem to agree on just how bad fits to the reduced residue function and to backward elastic scattering data are. For example, while Berger and Fox achieved a fair fit to π^-p backward scattering using an elastic $\Delta(1238)$ width a factor of two too small, Fenster and Wali (1970) used the correct $\Delta(1238)$ width as input and found a $d\sigma/du$ for π^-p at $p_{lab}=9.9$ BeV/c, a factor of 2000 too large. Generally, Regge fits to π^-p backward scattering have been rather poor, due to the violent change in the effective experimental reduced residue from $u=0$ to the $\Delta(1238)$ position. Examples of such fits may be found in Barger (1969b).

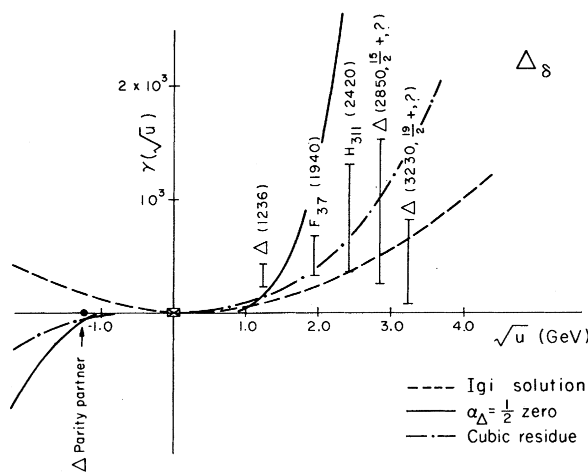

FIG. 10.12. Reduced residue function for Δ_δ trajectory in πN scattering. The phenomenological reduced residues were computed in terms of total widths and masses of resonances setting the scale parameter s_0 equal to the inverse of the trajectory slope. The size of the brackets comes from varying s_0 between 0.9 and 1.0 (GeV)^2 and moving the resonance positions between the values $M_{res}\pm\Gamma_{res/4}$. The resonance parameters are taken from Rosenfeld *et al** (1969). The x at $u^{1/2}=0$ indicates the value of the reduced residue obtained by Barger and Cline (1968) from π^-p backward elastic scattering fits. The dashed curve is taken from Igi (1968). The dot-dashed and solid curves result from two possible Veneziano parameterizations of Berger and Fox (1969).

comment briefly on this situation and refer the reader to Berger and Fox for further details.

The experimental data suggest that the meson baryon nonspin flip amplitudes A' have a residue zero at $t=-0.2$ (GeV)^2 associated with the ρ, ω, and A_2 quantum numbers (Dolen, Horn, and Schmid, 1968; Rarita *et al*, 1968; Michael and Dass, 1968; Dass, Michael, and Phillips, 1969). The evidence for the "crossover" phenomenon comes from pp, πp, and Kp elastic scattering.

Furthermore, the data for $\pi^-p\rightarrow\pi^0n$ suggests that the $B^{(-)}$ amplitude has a zero at $\alpha_\rho(t_0)=0$ [$t_0\cong-0.6$ (GeV)^2]. Now if the system is exchange degenerate ($\alpha_\rho=\alpha_f=\alpha_\omega=\alpha_{A_2}=\alpha$), all the A' and B amplitudes should have a residue zero at $\alpha=0$. This is easy to guarantee in the narrow resonance model because there is a convenient factor $\Gamma^{-1}(\alpha)$ present. Unfortunately the zero in $A'^{(-)}$ at $t=-0.6$ (GeV)^2 is not observed.

Similarly, the observed zero in A' at $t=-0.2$ (GeV)^2 associated with the ω implies by exchange degeneracy an accompanying one in β_f. This is attractive for several reasons: it allows one to explain the lack of shrinkage in scattering as due to a sign change in the f-Pomeranchon interference term at $t=-0.2$ (GeV)^2 and it is consistent with the duality arguments of Dolen, Horn, and Schmid (1968), who associate the crossover zero with the zeros of the Legendre polynomials of the prominent s-channel resonances in πp and Kp scattering. The unfortunate difficulty with this solution is that, by factorization, it leads to an unobserved zero in the

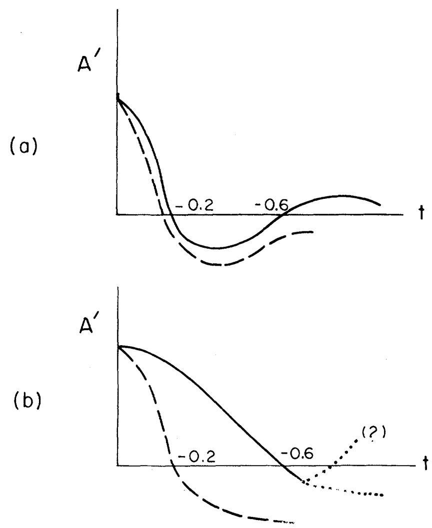

FIG. 10.13. Two possible theoretical forms of the A' amplitude (solid line) in meson baryon scattering. Situation (a), in which one predicts both the "crossover" zero at $t=-0.2$ (BeV)^2 and another zero at $t=-0.6$ (BeV)^2, is forced by duality. Situation (b) arises because of factorization. The dashed curves represent the result of including cut contributions or secondary trajectories in order to obtain agreement with experiment.

B amplitude at $t=-0.2$ (GeV)^2. The absence of these zeros is commonly attributed to the presence of cuts, which destroy factorization. The situation is illustrated in Fig. 10.13.

As we have emphasized above, the Pomeranchon can be included here only at the expense of also having exotic resonances (Wong, 1969a). Berger and Fox try to do this, but their results do not convince the authors that this is the way things work. One is forced to try to fit with a high-slope Pomeranchuk trajectory which tends to generate too much shrinkage and does not fit differential cross sections well. In πp scattering, Berger and Fox were able to get a reasonable fit with a high-slope Pomeranchon. This is shown in Fig. 10.14. Difficulties arise for $K^{\pm}p$ elastic scattering. There is too much shrinkage and $(d\sigma/dt)(K^-p)-(d\sigma/dt)(K^+p)$ is not well reproduced. This is shown in Fig. 10.15 for a Pomeranchon slope of $\alpha_P'=0.85$.

In conclusion, it is not possible to fit meson–baryon elastic scattering data in detail with a simple sum of Veneziano terms. Furthermore, even if one is allowed the freedom of employing arbitrary numbers of such terms, a really satisfactory fit has not been obtained. This is probably due to the presence of contributions from J-plane cuts which are not included in narrow resonance models.

Note added in proof: The reader may find it interesting, in connection with four-point narrow resonance phenomenology, to examine the recent work of Odorico (1971a). He suggests that the data for $K\bar{p}$ elastic and charge exchange scatterings shows dips in the amplitudes, which appear in a pattern of roughly parallel straight lines, much as do the zeros of the narrow resonance model.

E. Phenomenological Applications of Narrow Resonance N-Point Functions

Since the extension of Veneziano's original four-point construction to the N-point case discussed above in Sec. VII.C, a large volume of work has been devoted to analyzing experimental production data in this framework. In order to assess the significance of these efforts, it is necessary to balance whatever successes in fitting they entail against the uncertainties, assumptions, and arbitrary choices to which one is forced in order to construct a phenomenological multiparticle amplitude of the narrow resonance type.

Typically an attempt is made to fit the following features of data:

(1) Energy dependence and absolute magnitude of cross sections, perhaps of several processes related by crossing;
(2) Center of mass angular distributions of individual particles;
(3) Invariant mass distributions for two or more particles, including amounts of resonance production;
(4) Decay correlations of the dominant (low-lying) resonances.

Whatever successes one obtains here must be weighed against the following problems:

(1) Unitarity must be approximately imposed. This is usually done by making the trajectory functions complex, as in (10.5), and introducing a few parameters for each trajectory. These parameters can in principle be determined from sources independent of the particular processes being studied.
(2) One must select the narrow resonance N-point functions most likely to be important from the numerous (12 for $N=5$) nonequivalent amplitudes corresponding to different orderings of the external lines.
(3) Those amplitudes whose orderings of external lines give rise to exotic resonances are arbitrarily excluded.
(4) Particular Regge trajectories, out of all those allowed by the quantum numbers of the reactions under consideration, are marked as important and included in the phenomenological amplitude.
(5) The determination of the relative strengths and phases of the few amplitudes that are retained is generally arbitrary, although there may be some evidence that can guide the choices. *A posteriori* justification is provided by successes and/or failures, but little or no exploration has been made of the sensitivity of the fits to variations in the relative strengths of different amplitudes.
(6) Some plausible *ansatz* must be made in order to simulate effects of baryon spin. The assumptions put in here have influence on the low-mass regions of invariant

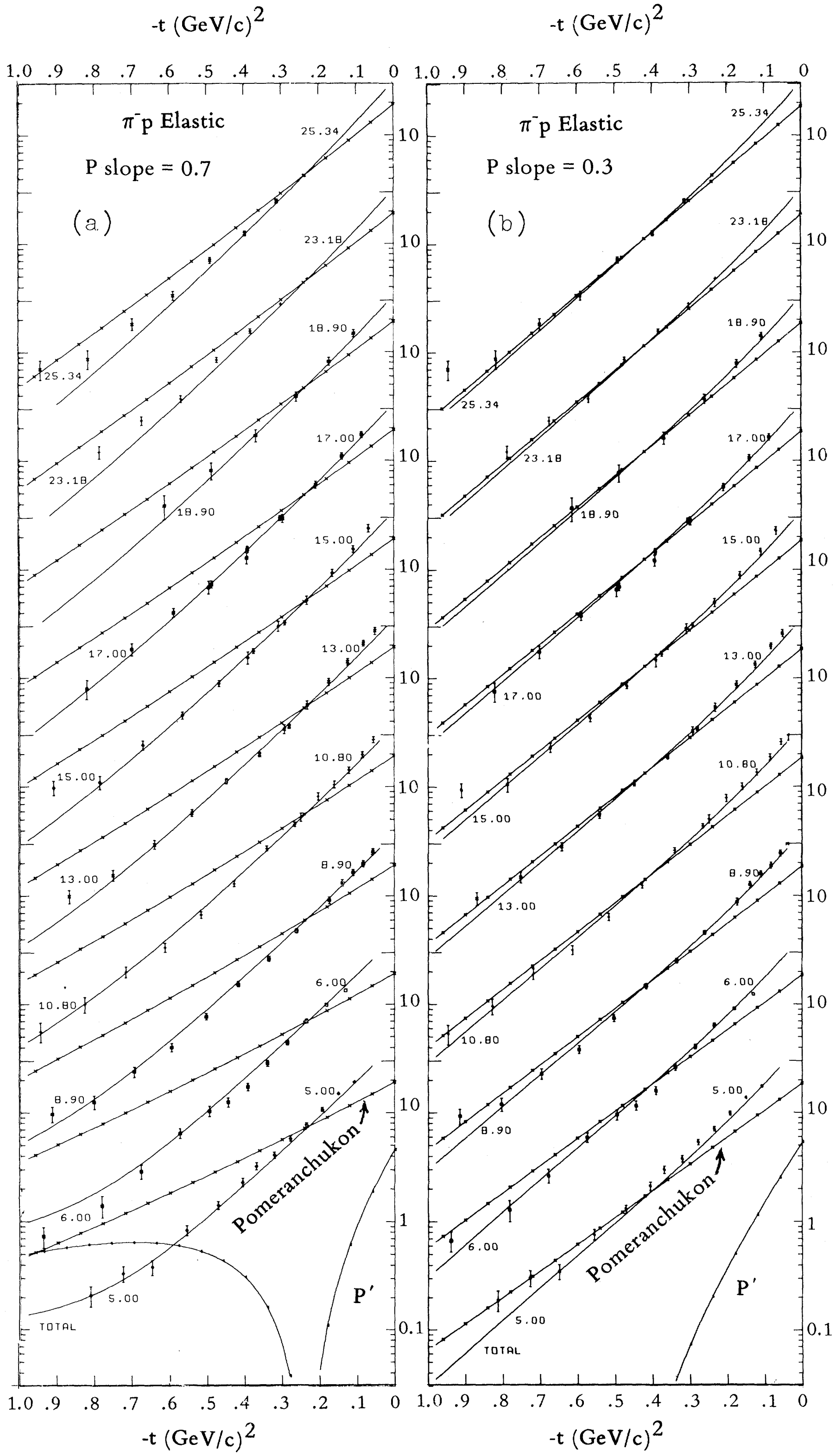

FIG. 10.14. Classical Regge pole fits to $\pi^- p$ elastic $d\sigma/dt$ data by Berger and Fox (1969) showing effect of using high-slope Pomeranchon. The total $(P+P'+\rho+\rho')$ contribution is the plain solid line. Contribution of the Pomeranchon alone is the solid line with x's. The P' contribution is given at the lowest energy in order to show how its residue zero moves as $\alpha_{P'}$ is altered. Data from Coffin *et al.* (1967) and Foley *et al.* (1963, 1965). More details can be found in Berger and Fox (1969).

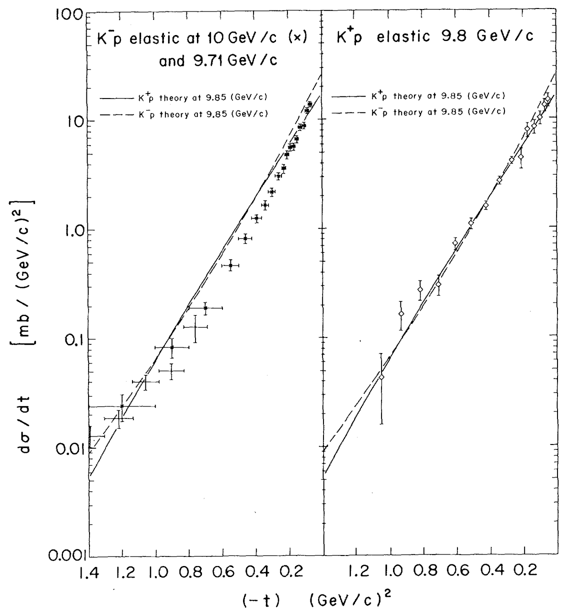

FIG. 10.15. Data for $K^{\pm}p$ elastic scattering, taken from Foley *et al.* (1963), Orear *et al.* (1968), and Aachen–Berlin–CERN–Imperial College–Vienna collaboration (1967).

mass distributions and also on momentum transfer distributions at small t.

In comparing the two lists above, the reader will notice that some assumptions included in the problem list more or less directly determine the experimental feature one is trying to describe. For example, it has been known for some time that $K^*(890)$ production with no transfer of charge in the t channel is dominated by natural parity exchanges (Gottfried and Jackson, 1964; Jackson, 1964). Therefore, if one attempts a B_5 description of K^* production and if one assumes that the dominant exchange is that of a vector meson trajectory, the "predicted" and observed K^* decay correlations are bound to agree. Similarly, if the imaginary part of a meson or baryon trajectory has been chosen to fit the width of the lowest-lying resonance, the shape of that particular resonance in the invariant mass distributions will be in reasonable accord with the data.

In other words, evaluation of the significance of fits with multiparticle dual amplitudes necessitates careful study to find those features intrinsic to the model, and not just properties which would emerge from any model that includes the dominant resonances and the peripheral nature of Regge pole exchange. In this connection it would be interesting to compare a narrow resonance fit with a crude coherent phenomenological fit using Breit–Wigner forms.

An example of an attempt to fit several reactions with three-body final states, using B_5 functions, is the work of Chan, Raitio, Thomas, and Tornqvist (1970). They study the three processes: (A) $K^+p \to K^0\pi^+p$, (B) $K^-p \to \bar{K}^0\pi^-p$, and (C) $\pi^-p \to K^0K^-p$, all of which are related by crossing. The comparison of the Chan *et al.* phenomenological forms with data is essentially in four parts: (1) energy dependence of total cross sections; (2) angular distributions; (3) mass spectra; (4) momentum transfer distributions (in parallel with our list of possible predictions above). Using the same narrow resonance formula and one free parameter, Chan *et al.* attempt to describe the above features of (A)–(C) over an energy range of 2.5–13 GeV. As we shall see, the presence of the problems listed above makes it difficult to identify any features of the data which are intrinsic to the narrow resonance model.

Let us begin with the predictions of the high-energy behavior of total cross sections.

At high energies, say $p_{\mathrm{Lab}} > 4$ BeV, the model is constructed to be doubly peripheral. Once vector meson exchange is input, as it is by Chan *et al.*, the amplitude will Reggeize with the vector meson trajectory dominating and the high-energy dependence will be correct. At low energies, the predictions badly undershoot the data for (A) and (B), and the shape, though not the normalization, agrees for (C). At low energies one expects kinematic factors due to spin and phase space to become important. The phase space factor will be especially significant for the heavy final state in (C), while spin factors are expected to play an important role in (A) and (B). According to Berger (1969d), if one extrapolates to low energies, phenomenological four-point fits at high energies, using the asymptotic form of the kinematic spin factors, one badly undershoots the data, just as here. Since the phase space factor is expected to dominate at low energies in (C), the shape agreement there is not surprising.

Chan *et al.* further achieve a rough agreement (within a factor of 2) for the relative normalizations of (A)–(C). This agreement can be understood using relatively simple arguments contained in, but more basic than, the B_5 approach.

The total cross sections for (A) and (B) versus (C) involve a factor of $\approx 1/20$. A crude argument yielding an order of magnitude effect of this kind is as follows: Suppose (A) and (B) proceed via $Kp \to K^*p$ and $K^* \to K\pi$, while (C) proceeds via $\pi p \to A_2p$ and $A_2 \to K\bar{K}$. Then we crudely expect σ_A/σ_C to be proportional to $\Gamma(A_2 \to \bar{K}K)/\Gamma(A_2 \to \pi\rho) \approx 1/10$, since the reactions are otherwise similar. In the work under consideration vector exchange dominates, a total A_2 width of 90 MeV is input, and presumably one is correctly taking account of angular momentum barrier and phase space effects, so that $A_2 \to \bar{K}K$ will be properly suppressed with respect to $K^* \to K\pi$.

With respect to the mass spectra, as we have mentioned above, generally the prominent resonance on each relevant trajectory is treated correctly. On the other hand, the second resonance on each leading trajectory is not well predicted. In fact, if one examines the many plots of mass spectra given by Chan *et al.*, one discovers that the secondary structure of the theoretical curves does not match that of the data, to the extent that there are several unobserved resonances in the theoretical curves which necessitate special discussion. In making the fits, we should also point out that each of the six relevant trajectories ($\omega - A_2$, K^*, N_α, Δ, Λ, Y_1^*) is made complex as in item (1) of our list of problems above. Each trajectory is parameterized with five constants, making a total of 30 parameters roughly fitted with the known masses and widths. In these circumstances there is a subtle question which needs to be asked as to whether or not the parameters are really being fixed independently of the data.

With respect to the angular distribution comparisons, we have the following comments. In the reactions (A) and (B), the predicted angular distribution for $K^{*\pm}$ production are quite reasonable as are the Δ^{++} distributions in (A). However, this has nothing to do with the details of the narrow resonance model, but in fact follows from the Gottfried–Jackson theorem (Gottfried and Jackson, 1964), which gives precisely these distributions for vector meson and Δ production through vector meson exchange. [The Jackson and Treiman–Yang distributions are $\sin^2\theta$ and $1-\cos 2\varphi$, respectively. As in item (6) of the problem list above, the external baryons are considered to be spinless here. Otherwise the statement about Δ production would be incorrect.]

An interesting discrepancy appears in the Jackson angle distributions in K^{**} production in (A). Though the forward distribution roughly fits data, there is a disagreement at backward angles which becomes progressively worse at higher energies. The agreement at forward angles is due to an input choice of constructive interference between the Δ and K^{**} bands in the Dalitz plot. The disagreement at backward angles is a real failure of the model, but it is difficult to pin down the cause without detailed analysis. The simplest possibility would be that one is seeing the unobserved 1^- state in the $K^{**}(1420)$ tower. However, there are complicated coherent interference effects and reflections across the Dalitz plot, as one would expect with the simultaneous presence of degenerate resonance towers in several channels, so a definitive statement on this point cannot be made.

Last, we come to the distributions in momentum

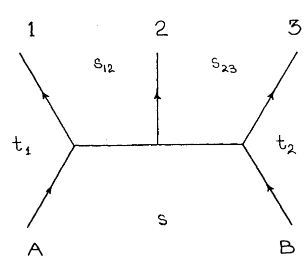

FIG. 10.16. Kinematics for five-point amplitude.

transfer squared, t. As noted explicitly by Chan *et al.*, neglect of the external nucleon spin eliminates from consideration any possible non-spin-flip amplitude. Such an amplitude would make a nonzero contribution at $t=0$, where the spin-flip amplitude vanishes. The t distributions given by Chan *et al.* agree with data for large t, but there are discrepancies at small t which probably arise from a non-spin-flip contribution. The magnitude of this contribution remains an open question, and should be further studied.[60] To close this section, we would like to make some brief general remarks about some further aspects of the five point problem.

First of all, we define variables as in Fig. 10.16 for the process $AB\rightarrow 123$, all particles being scalars. Experimentally (Bartsch *et al.*, 1968; Oh and Walker, 1969), it is observed that one can parameterize the doubly differential cross-section for $AB\rightarrow 1+N$, N being a collection of N hadrons, by

$$d^2\sigma/ds_N dt = A(s_N)\,\exp[-b(s_N)\,|t-t_0|] \quad (10.18)$$

near $t=t_0$ and for small s_N, where $s_N=p_N^2$, $t=(p_A-p_1)^2$, and t_0 is the forward limit of t. Empirically, $b(s_N)$ has little or no resonance structure and is a monotonically decreasing function, while $A(s_N)$ shows the effect of resonances.

In the five-point case, Jones and Wyld (1969a) have made the interesting observation that even if the subenergy, s_{23}, is small, the Bardakci–Ruegg function B_5 yields, for large $s=(p_A+p_B)^2$, a smooth and monotonically decreasing $b(s_{23})$. As shown by the following argument, due to Berger (1969a, d), this follows from the multiperipheral nature of the sum over Feynman tree graphs from which B_5 is constructed. For s large with respect to the inverse of the universal slope, B_5 takes the limit (Bialas and Pokorski, 1969)

$$B_5 \sim s^{\alpha(t_1)}\Gamma(-\alpha(t_1))B[\alpha(s_{23}),\alpha(t_2)]$$

$$\times {}_2F_1[-\alpha(t_1),-\alpha(s_{23});-\alpha(t_2)-\alpha(s_{23});s_{13}/s], \quad (10.19)$$

where s_{23} and t_i are fixed, s_{12} is large, and B is the ordinary beta function. The first two factors in this expression yield the usual exponential forward peak in t_1, while the beta function does the same for the distribution in t_2. The hypergeometric function is slowly varying over the kinematic region of interest and essentially plays no role. This means that B_5 approximately factors into the form $f_1(t_1)\cdot f_2(t_2)$, with the f_i dropping exponentially with increasing argument. Using a straightforward phase space argument (Berger, 1969a, d), it can then be shown that the resultant doubly differential cross section has the required behavior (10.18).

Jones and Wyld (1969b) have also examined the

[60] One of us (J.Y.) would like to thank E. L. Berger for an extensive and informative discussion of his work, which resulted in the remarks above. We would also like to thank V. Walach for many helpful discussions and access to his data.

problem of fitting the $\bar{p}n \to 3\pi$ data using functions of the B_5 type, rather than the B_4's of Lovelace, Berger, and Altarelli and Rubinstein, discussed above. Neglecting the nucleon spins,[54] they find that no reasonable fit to the data is possible if one inserts the measured parameters of the ρ and f. The experimental fits (Anninos *et al.*, 1968; Foster *et al.*, 1968) lead one to believe the data cannot be fitted with real ρ and f parameters unless some complicated interference occurs. The function B_5 is complicated but evidently not in the correct manner.

Additional material concerning narrow resonance phenomenology can be found in Bose and Gupta (1969), Capella *et al.* (1969), Gunion and Yesian (1969), Gutay *et al.* (1969), Pinsky (1969), Roberts (1969), and Moen and Moffat (1970).

XI. CONCLUSION

Work on narrow resonance models can conveniently be split into three stages: the breakin four-point stage, the dog fight N-point stage, and the breakout or unitarization stage.[61]

At the four-point and N-point stages, the model is, even though physically inapplicable, rather simple and beautiful. The N-point amplitudes can be characterized as functions having the singularity structure of Feynman tree graphs and possessing multi-Regge limits. Evidently, provided we also assume asymptotic exponential fall-off when subenergies in which resonances are absent are held fixed, there are general uniqueness statements which can be made.[62] The general properties of N-point functions which are forced to have tree graph singularity structure and multi-Regge behavior have not been fully elucidated and deserve further investigation.

The following questions regarding four-point functions also seem to us to merit further study:

(a) Can uniqueness, in the $\pi\pi$ problem, be rigorously related to the positivity of resonance widths?

(b) Is it possible to find a general, simple, way to parametrize four-point amplitudes in a narrow resonance manner, for processes with arbitrary external spins, even if one requires the elimination of exotic trajectories?

(c) Beginning with a particular $\pi\pi$ amplitude and making all internal poles scattering states, and vice versa, can one find a closed selfconsistent set of four-point amplitudes?

(d) Is it possible to prove rigorously that nonlinear narrow resonance mass formulas necessarily do not lead to full Regge behavior?

(e) In a narrow resonance model for baryons, is it possible to escape parity doubling? (Carlitz and Kislinger, 1970).

As we discussed above in Sec. VII for the N-point functions so far invented, it is possible to force factorization, provided one is willing to accept degeneracy of satellite trajectories. The minimal such degeneracy seems to be that of the statistical model of Hagedorn (1968), and necessarily seems to involve ghosts, which can be associated, via the harmonic oscillator operator formalism (Fubini, Gordon, and Veneziano, 1969) with the appearance of an indefinite metric.

The role of internal symmetry in the N-point narrow resonance model is so far ill understood. In Sec. VII we have discussed straightforward attempts to combine the narrow resonance model with the quark model, assuming that amplitudes can be separately factorized into orbital, spin, and internal symmetry parts. This seems unsatisfactory and it would be interesting to know if the narrow resonance model really forces amplitudes to have this artificial decomposition, which results in ghosts, parity doubling, and extra unobserved trajectories, all associated with the spin piece of the factorization.

With respect to the narrow resonance N-point functions, the following questions seem to us of interest:

(a) Can one find a narrow resonance N-point bootstrap consistent with the Goldstone realization of $SU(2)\otimes SU(2)$, with $m_\pi=0$ and $m_\rho > m_\pi$?[63]

(b) In what sense is the N-point model unique?

(c) Can one find a solution to the inside–outside four-point question (c) above, valid for N-point functions?

(d) Is there some general rule which tells us how zeros of narrow resonance amplitudes behave, as unitarity is implemented?

Though narrow resonance amplitudes are conceptually extremely useful, there are general problems with using them phenomenologically. These model amplitudes have linear trajectories with a universal slope, in agreement with the empirical result that known Regge trajectories appear to be approximately linear with slopes of around 0.9–1.0 $(\mathrm{GeV})^{-2}$. However, the satellite trajectories in the model do not correspond, in even a rough way, to anything anyone has so far observed. Furthermore, if one attempts to force a narrow resonance parameterization, say, to fit data for meson–nucleon scattering, the resulting expressions become prohibitively complicated, no less so than the final expressions in classical Regge pole fits. In fact,

[61] Field Marshal Montgomery, Earl of Alamein, *El Alamein to the River Sangro* (Hutchinson, London, 1956), pp. 13, 16 ff. One of us (J.Y.) would like to thank Professor Y. Ne'eman for suggesting this analogy with desert warfare.

[62] In this connection see Khuri (1969) and also the recent work of Tiktopoulos (1970).

[63] See for example the work of Kernan and Shepard (1969) on $\pi^- p \to \pi^+ \Delta^-$.

since a narrow resonance parameterization is the extreme case of a pure moving Regge pole model, the difficulties associated with such classical Regge fits are made even more evident in this context. This is in accord with recent suggestions (see especially Fox, 1969) that cuts in the J-plane are present and empirically significant.

If one is less ambitious, there are several interesting features of narrow resonance models that can be separately compared against experiment.

The first of these is the general question of the validity of local duality, which would imply, along a direction in the Mandelstam plane corresponding to an exotic channel, an oscillation of the amplitude, due to the cancellation of exchange degenerate trajectories in other channels.[64] There is as yet no definitive test.

Secondly, N-point functions can be represented as a sum over Feynman tree graphs and they therefore acquire a multi-Regge asymptotic behavior with a nontrivial dependence on the Toller angle, arising from the structure of the model Reggeon–Reggeon-resonance vertex. So far this dependence has not been checked experimentally; it would be interesting to do so.

Thirdly, several models beside the narrow resonance model predict the existence of secondary trajectories, and to test for these one must devise a way of performing detailed partial wave analyses at medium energies in order to find out whether or not known resonances contain several resonating components of different spins.

Fourthly, in the narrow resonance model, trajectories *effectively* become linear in mass rather than $(\text{mass})^2$ as one goes to higher energies, and it would be very interesting to see whether or where linear $(\text{mass})^2$–spin relations break down experimentally.

Lastly, there is an interesting test of one basic feature of the narrow resonance model that actually works. If the inverse slope of Regge trajectories, b^{-1}, is actually a universal scale parameter as in the model, one would expect that the observed slope of diffraction peaks in πN and NN high-energy charge exchange scattering could be roughly computed by keeping only the leading Regge trajectory. This turns out to be true (Shapiro and Yellin, 1970 remark C). In fact, this is a specialization of the general observation of Veneziano (1968) that both narrow resonance amplitudes and data fall exponentially for fixed $\cos\theta$ (Orear, 1964).

From the point of view of the authors, the breakout stage has not occurred as yet. Thus far, attempts to unitarize have gone in two directions: *ad hoc* modification of the original formulas, and utilization of the factorization properties of the N-point functions to generate a perturbation series with iterative unitarity, of course including closed loops.

[64] We thank G. F. Chew for emphasizing the importance of this point to us.

The *ad hoc* modifications seem doomed to failure precisely because they are *ad hoc*, and the physical complications involved in constructing unitary amplitudes due to the nonlinear nature of the unitarity equations and the infinity of inelastic channels which couple through unitarity seem to us to require a more physical and systematic approach.

It is hard to visualize how the perturbation approach will cure the unphysical pathologies of the original N-point narrow resonance amplitudes, unless each iteration produces very large corrections. After renormalization, the satellite trajectories must plunge into distant regions of the complex-J plane while leaving the leading trajectories with reasonable properties. The necessity for large corrections at each stage of the iteration procedure creates a danger that the procedure will not be stable and will not converge to a well-defined answer. Because of the nature of the problem, the perturbation series approach is going to be investigated no matter how remote the possibility of success. We therefore prefer to maintain an attitude of contemplative but extreme skepticism. We would like, however, to emphasize in this connection that the following three questions require answers: (a) Are cross-sections positive in each iterative order?[65] (b) Is there any qualitative argument which would lead one to believe that the leading and satellite trajectories behave as suggested above? (c) In what sense and at what iterative order will duality be broken?

The basis of our skepticism *vis-à-vis* the iterative approach is that, at least in the planar graph case, duality is being preserved at each stage of the iterative procedure in the graphical sense of Fig. 9.3. Since exact duality seems to be in conflict with experiment, one might suppose that the breakout stage will be associated with a physical principle which tells us how duality is broken and at the same time generates Regge cuts, Pomeranchon effects, and exotic resonances.

One of us (Yellin, 1969d) has recently suggested a way to interpret the narrow resonance scheme which is essentially orthogonal to that of the iterators and which includes a duality-breaking mechanism. In this approach, one supposes that the hadrons are built out of quarks interacting through the exchange of an equally fictitious harmonic oscillator quantum, the oscillon. (This type of interaction is chosen in order to have infinitely rising trajectories in the narrow resonance limit.) The narrow resonance model then consists of amplitudes in which quarks interact in a relativistic "potential," with closed quark loops being forbidden, and the narrow resonance poles are to be treated as bound states. In this approach, it is evident that what-

[65] It is not known whether the various ghosts in the iterative approach lead to negative cross sections in every iterative order. Some problems in this connection have been discussed by Lee and Wick (1968) and Lee (1969).

ever the 0th order amplitude may be, it is not a Born term, containing as it does all orders in the quark–quark–oscillon coupling. Furthermore, if one generates bound states by summing all crossed ladders in the oscillon model, there is no reason to expect that all couplings of the resultant bound states to each other will factorize without the introduction of additional degeneracies.

If one takes the QED analogy somewhat seriously, the last point could conceivably be checked. One is instructed to take the $2N$-point function for N electrons and N positrons, with only multiphoton exchange, and make all couplings of positron bound states to each other factorize. While the complete theory, including closed loops, will factorize perfectly, one would suspect that in this truncated version it is inappropriate to attempt to factorize, and the result of forcing factorization will, at the minimum, lead to a large but finite degeneracy, just as in the narrow resonance bootstrap.

There is a natural way of introducing duality breaking into such a scheme. We merely start adding in diagrams with closed quark loops, which, according to conventional wisdom (Mandelstam, 1963a, b, c), bring in asymptotic behavior typical of cut structure in the J plane. Though this picture is helpful in guiding one's mind towards a workable alternative to the iterative approach, it has the failing that rules for computing anything do not yet exist.

We conclude with the following[66]:

The whole process is a lie
  unless,
    crowned by excess,
it break forcefully,
  one way or another,
    from its confinement—

***

We will it so
  and so it is
    past all accident.

ACKNOWLEDGMENTS

The authors would like to thank the following people for helpful comments and/or suggestions: K. Bardakci, M. A. B. Beg, E. Berger, R. Brower, G. F. Chew, J. A. Cronin, R. F. Dashen, S. Fenster, F. Gilman, J. D. Jackson, N. N. Khuri, C. Lovelace, S. Mandelstam, J. Mandula, G. Shaw, E. P. Tryon, C. Thorn, G. Veneziano, S. Weinberg, J. Weis, and G. Zweig.

The following people were kind enough to read portions of the manuscript before final preparation: D. Austin, R. Brower, R. Cahn, R. F. Dashen, S. Mandelstam, J. Mandula, C. Sorensen, and J. Weis.

We would also like to thank Mmes. E. Gorman, C. Graham, P. Harris, S. Mennella, and G. Perry for their patient help in preparing the manuscript.

One of us (J. Y.) would like to thank Goeffrey Chew for his hospitality at Lawrence Radiation Laboratory, Berkeley, where this work was begun.

Finally, we would like to thank C. Cronström and J. D. Jackson for proofreading and commenting on the final complete version.

[66] W. C. Williams, "The Ivy Crown," in *Pictures from Brueghel* (New Directions, New York, 1962), p. 124.

LECTURE NOTES AND REVIEW ARTICLES

Bassetto, A., 1970, Fortschr. Physik **18**, 185.
Chan, H. M., 1969, Report to the Royal Society, CERN Preprint (to be published).
Diebold, R., 1969, Proceedings Boulder Conference on High Energy Physics, (Colorado Assoc. U. Press, Denver, 1970).
Fubini, S., 1969, Comments Nucl. Particle Phys. **3**, 22.
Greenberg, O. W., 1969, Proc. Lund International Conf. on Elementary Particles (Berlingska Boktryckeriat, Lund, 1969).
Harari, H., 1968, Proc. Ann. Intern. Conf. High Energy Phys. Vienna **14**, 195.
——, 1969, Paper presented at Liverpool Conference (to be published).
Horn, D., 1966, "Lectures on $SU(6)_W$," Argonne National Laboratory Report (unpublished).
Jackson, J. D., 1970, Rev. Mod. Phys. **42**, 12.
Jacob, M., 1969, Acta Phys. Austriaca Suppl. **6**, 215.
Mandelstam, S., 1969, Comments Nucl. Particle Phys. **3**, 65.
Morpurgo, G., 1968, Proc. Ann. Intern. Conf. High Energy Phys. Vienna **14**, 225.
Predazzi, E., 1969, Lecture Notes, Boulder Summer School (to be published).
Risk, C., 1968, "High-Energy Behavior of Feynman Diagram," Michigan Preprint (unpublished).
Rosenfeld, A., N. Barash-Schmidt, A. Barbaro-Galtieri, L. Price, P. Soding, C. Wohl, M. Roos, and G. Conforto, 1969, Rev. Mod. Phys. **41**, 109.
Tripp, R., 1969 Hawaii Topical Conf. on Particle Physics (Western Periodicals, Los Angeles, 1970).
Veneziano, G., 1969 Veneziano: Proc. Coral Gables Conf. on Fundamental Interactions at High Energies (Gordon and Breach, New York, 1970).
Weinberg, S., 1969, Comments Nucl. Particle Phys. **3**, 28.
Wilson, R., 1966, *Particle Interactions at High Energies* (Oliver and Boyd, Edinburgh).

BIBLIOGRAPHY

Books (Physics)

Adler, S., and R. Dashen, 1968, *Current Algebras* (Benjamin, New York).
Bjorken, J. D. and S. D. Drell, 1964, *Relativistic Quantum Mechanics* (McGraw-Hill, New York).
Chew, G. F., 1961, *S Matrix Theory of Strong Interactions* (Benjamin, New York).
——, 1964, *The Analytic S Matrix.* (Benjamin, New York).
Collins, P. D. B., and E. Squires, 1968, *Regge Poles in Particle Physics* (Springer, New York).
de Alfaro, V., and T. Regge, 1965, *Potential Scattering* (North-Holland, Amsterdam).
Eden, R. J., 1967, *High Energy Collisions of Elementary Particles* (Cambridge U. P., New York).
——, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, 1966, *The Analytic S-Matrix* (Cambridge U. P., New York).
Frautschi, S., 1963, *Regge Poles and S-Matrix Theory* (Benjamin, New York).
Jacob, M., and G. F. Chew, 1964, *Strong-Interaction Physics* (Benjamin, New York).
Newton, R. G., 1964, *The Complex J-Plane* (Benjamin, New York).

Omnes, R., and M. Froissart, 1963, *Mandelstam Theory and Regge Poles* (Benjamin, New York).
Squires, E. J., 1964, *Complex Angular Momenta and Particle Physics* (Benjamin, New York).

Books (Mathematics)

Ahlfors, L. V., 1966, *Complex Analysis* (McGraw-Hill, New York), 2nd ed.
Akhiezer, N., and M. Krein, 1938, *Some Questions in the Theory of Moments* [Russian original; Kharkov, 1938 (Engl. transl.: American Mathematical Society, Providence, Rhode Island, 1962)].
American Mathematical Society, 1966: Proceedings of Symposia in Pure Mathematics XI: *Entire Functions and Related Parts of Analysis* (American Mathematical Society, Providence, R. I.).
Bers, L., 1963, "Introduction to Several Complex Variables," Lecture Notes, Courant Institute, New York University (unpublished).
Buchholz, H., 1969, *The Confluent Hypergeometric Function* (Springer, New York).
Caratheodory, C., 1954, *Theory of Functions* (Chelsea, New York), Vols. 1 and 2.
Corput, J. G. van der, 1954, "Asymptotic Expansions I, II, III," Lecture Notes, Department of Mathematics, University of California, Berkeley (unpublished).
Erdelyi, A., *et al.*, 1954, *Higher Transcendental Functions* (McGraw-Hill, New York), Vols. 1–3; referred to in text as EHF.
Erdelyi, A., *et al.*, 1954, *Tables of Integral Transforms* (McGraw-Hill, New York), Vols. 1 and 2; referred to in text as EIT.
Fuks, B. A., 1963, *Introduction to the Theory of Analytic Functions of Several Complex Variables* [Russian original: Moscow, 1962 (Engl. transl.: American Mathematical Society, Providence, R. I.)].
Gradshteyn, I. S., and I. M. Ryzhik, 1965, *Tables of Integrals, Series, and Products* (Academic, New York); referred to in text as GR.
Hörmander, L., 1966, *An Introduction to Complex Analysis in Several Variables* (Van Nostrand, Princeton).
Jordan, C., 1947, *Calculus of Finite Differences* (Chelsea, New York).
Marden, M., 1949, *Geometry of the Zeros of a Polynomial in a Complex Variable* (American Mathematical Society, Providence, R. I.).
Markushevich, A. I., 1966, *Entire Functions* [Russian original: Moscow, 1965 (Engl. transl.: Elsevier, New York)].
Nielsen, N., 1906, *Handbuch der Theorie der Gammafunktion* (Teubner, Leipzig).
Nielsen, N., 1923, *Traite Elementair des Nombres de Bernoulli* (Gauthier-Villars, Paris).
Norlund, N., 1924, *Vorlesungen uber Differenzenrechnung* (J. Springer, Berlin).
Siegel, C. L., 1948, "Analytic Functions of Several Complex Variables," Lecture Notes, Institute for Advanced Study, Princeton (unpublished).
Szego, G., 1939, *Orthogonal Polynomials* (American Mathematical Society, Providence, R. I.).
Titchmarsh, E. C., 1939, *Theory of Functions* (Oxford U. P., London, England), 2nd ed.
Watson, G. N., 1966, *Bessel Functions* (Cambridge U. P., London, England), 2nd ed.
Whittaker, E. T., and G. N. Watson, 1927, *Modern Analysis* (Cambridge U. P., London, England), 4th ed.

REFERENCES

References are followed by roman numerals indicating the sections in this review in which they are cited.

Aachen–Berlin–CERN–Imperial College–Vienna Collaboration, 1967, Phys. Letters **24B**, 434 (X).
Abarbanel, H. D. I., 1970, Ann. Phys. (N.Y.), **57**, 525 (II, VII).
——, and C. Itzykson, 1969, Phys. Rev. Letters **23**, 53 (VII).
Abers, E., and V. Teplitz, 1969, Phys. Rev. Letters **22**, 909 (V).
Abrams, G. S., *et al.*, 1968, Phys. Rev. Letters **21**, 1407 (X).
Ademollo, M., H. Rubenstein, G. Veneziano, and M. Virasoro, 1968, Phys. Rev. **176**, 1904 (II).
——, and E. del Giudice, 1969, Nuovo Cimento **63A**, 639 (V).
——, G. Veneziano, and S. Weinberg, 1969, Phys. Rev. Letters **22**, 83 (V).
Adler, S. L., 1965a, Phys. Rev. **137**, 1022; 1965b **140**, 736 (V, X).
——, 1966, Phys. Rev. **143**, 1144 (V).
Ahmad, M., Fayyazuddin, and Riazuddin, 1969, Phys. Rev. Letters **23**, 504; *ibid.* **23**, 1143 (E) (V).
Aks, S., 1965, J. Math. Phys. **6**, 515.
Allesandrini, V., D. Amati, and E. Squires, 1968, Phys. Letters **27B**, 463 (II).
——, and D. Amati, 1969, Phys. Letters **29B**, 193 (III).
Altarelli, G., and H. Rubenstein, 1969, Phys. Rev. **183**, 1469 (X).
Alwis, S. P. de, S. P. Nutbrown, P. Booker, J. Kosterlitz, 1969, Phys. Letters **29B**, 362 (V).
Amann, R. F., 1969, Nuovo Cimento Letters **2**, 87 (X).
Amati, D., M. Bellac, and D. Olive, 1969, Nuovo Cimento **66A**, 815. (VII, IX).
——, R. Jengo, H. Rubenstein, G. Veneziano, and M. Virasoro, 1968, Phys. Letters **27B**, 38 (V).
——, C. Bouchiat, and D. Gervais, 1969, Nuovo Cimento Letters **2**, 394 (IX).
Anderson, R. L., *et al.*, 1969, Phys. Rev. Letters **23**, 721 (V).
——, *et al.*, 1968, Phys. Rev. Letters **21**, 479 (V).
Anninos, P., L. Gray, P. Hagerty, T. Kalogeropoulos, S. Zenone, R. Bizzari, G. Ciapetti, M. Gaspero, I. Laakso, S. Lichtman, and G. C. Moneti, 1968, Phys. Rev. Letters **20**, 402 (X).
Antoniou, N. G., A. Bartl, and F. Widder 1969a b, Z. Phys. **228**, 143; *ibid.* **232**, 390 (III).
Arbab, F., and J. D. Jackson, 1968, Phys. Rev. **176**, 1796 (II).
——, 1969, Phys. Rev. **183**, 1207 (VI).
Argyres, E., and C. S. Lam, 1969, Phys. Rev. **186**, 1532 (IV).
Arnold, R. C., 1965, Phys. Rev. Letters **14**, 657 (III).
——, 1968, Phys. Rev. **153**, 1523 (X).
Arnowitt, R., P. Nath, Y. Srivastava, and M. H. Friedman, 1969, Phys. Rev. Letters **22**, 1158 (V).
Atkinson, D., L. A. P. Balázs, F. Galogero, P. DiVecchia, A. Grillo, and M. Lusignoli, 1969, Phys. Letters **29B**, 423 (VI).
——, and K. Dietz, 1969, Phys. Rev. **177**, 2579 (II).
Baker, W. F., *et al.*, 1968, Phys. Letters **28B**, 291 (X).
Balachandran, A., P. Freund, and C. Schumacher, 1964, Phys. Rev. Letters **12**, 209 (X).
——, 1969, Phys. Rev. **D1**, 2770 (II, IV).
Balazs, L. A. P., 1969, Phys. Letters **29B**, 228 (VI).
Bali, N. F., G. F. Chew, and A. Pignotti, 1967, Phys. Rev. Letters **19**, 614 (VII).
——, D. D. Coon, and J. W. Dash, 1969a, Phys. Rev. Letters **23**, 900 (VI).
——, D. D. Coon, and J. W. Dash, 1969b, University of Washington Preprint (unpublished) (III).
Banaigs, J., *et al.*, 1969, Nucl. Phys. **B9**, 640 (X).
Bander, M., 1969, Nucl. Phys. **B13**, 587 (V).
Bardakci, K., and H. Ruegg, 1968, Phys. Letters **28B**, 342 (VII).
——, 1969, (private communication) (X).
——, and M. Halpern, 1969, Phys. Rev. **183**, 1456 (VII).
——, M. Halpern, and J. Shapiro, 1969, Phys. Rev. **185**, 1910 (IX).
—, and S. Mandelstam, 1969, Phys. Rev. **184**, 1640 (V, VII, IX, X).
——, and H. Ruegg, 1969, Phys. Rev. **181**, 1884 (VII).
Barger, V. and D. Cline, 1966, Phys. Rev. Letters **16**, 913 (II).
——, and D. Cline, 1967, Phys. Rev. **155**, 1792 (II, X).
——, and D. Cline, 1968, Phys. Rev. Letters **21**, 392; **21**, 1132(E) (X).
——, and L. Durand, 1968, Phys. Letters **26B**, 588 (II).
——, and R. J. N. Phillips, 1968, Phys. Rev. Letters **21**, 865 (II, X).
——, 1969a, Phys. Rev. **179**, 1371 (X).
——, and D. Cline, 1969b, Phys. Rev. **182**, 1849 (X).
——, and C. Michael, 1969, Phys. Rev. Letters **22**, 1330 (X).
——, and R. Phillips, 1969, Phys. Letters **29B**, 676 (X).
Barshay, S., 1969, Phys. Letters **29B**, 365 (VII).
Bartsch, J., *et al.*, 1968, Phys. Letters **27B**, 336 (X).
Basdevant, J., and B. Lee, 1969, Phys. Letters **29B**, 437 (X).
Berger, E., 1969a, Phys. Rev. **179**, 1567 (X).
——, 1969b, Proceedings ANL Conference on $\pi\pi$ and $K\pi$ Interactions (X).
——, 1969c, Proceedings of the Irvine Conference on Regge Theory (to be published) (X).
——, 1969d (private communication) (X).

——, and G. Fox, 1969, Phys. Rev. **188**, 2120 (V, X).
Bialas, A., and S. Pokorski, 1969, Nucl. Phys. **B10**, 399 (VII, X).
Bitar, K., 1969a, Phys. Rev. **185**, 2032 (IV).
——, 1969b, Phys. Rev. **186**, 1424 (II, III).
——, and G. Tindle, 1968, Phys. Rev. **175**, 1835.
Bizzarri, R., 1968, Nuovo Cimento **53A**, 951 (X).
Bjorken, J., 1966, Phys. Rev. **148**, 1467 (V.C.).
——, 1969, Phys. Rev. **179**, 1547 (V.D.).
Blankenbecler, R., and M. Goldberger, 1962, Phys. Rev. **126**, 766 (III).
Boguta, J., 1969, Nucl. Phys. **B10**, 319 (III).
Bose, S. K., and K. C. Gupta, 1969, Phys. Rev. **184**, 1572 (X).
Botke, J. C., and R. Blankenbecler, 1969, Phys. Rev. **186**, 1536 (VI).
Brandt, R., 1969, Phys. Rev. **187**, 2192 (V).
Bronzan, J., I. Gerstein, B. Lee, and F. Low, 1967, Phys. Rev. Letters **18**, 32 (V).
Brower, R. C., and M. B. Halpern, 1969, Phys. Rev. **182**, 1779 (V).
——, A. Rabl, and J. Weis, 1970, Nuovo Cimento **65A**, 654 (V).
——, and J. Weis, 1969a, Phys. Rev. **188**, 2486; 1969b, **188**, 2495 (V).
Campbell, D. K., D. I. Olive, and W. J. Zakrzewski, 1969, Nucl. Phys. **B14**, 319 (VII).
Caneschi, L., A. Schwimmer, and G. Veneziano, 1969, Phys. Letters **30B**, 351 (VII, IX).
Capella, A., B. Diu, J. Kaplan, and D. Schiff, 1969, Nuovo Cimento Letters **1**, 665 (X).
Capps, R., 1969, Phys. Rev. **185**, 2008 (VIII).
Capra, F., 1969, Phys. Letters **30B**, 53 (II, IV).
Carlitz, R., and M. Kislinger, 1970, Phys. Rev. Letters **24**, 186 (II, XI).
Carrol, A., *et al.*, 1968, Phys. Rev. Letters **21**, 1282 (X).
Chan, H.-M., 1969, Phys. Letters **28B**, 425 (VII).
——, R. Raitio, G. Thomas, and N. Tornqvist, 1970, Nucl. Phys. **B19**, 173 (X).
——, and J. Paton, 1969, Nucl. Phys. **B10**, 516 (VII).
——, and S. T. Tsou, 1969, Phys. Letters **28B**, 485 (VII).
Chang, S.-J., R. Dashen, and L. O'Raifeartaigh, 1969, Phys. Rev. **182**, 1819 (V).
——, and S. Ma, 1969, Phys. Rev. **188**, 2385 (VII).
Cheng, H., and D. Sharp, 1963, Phys. Rev. **132**, 1854 (VI).
Chew, G. F., and S. Mandelstam, 1960, Phys. Rev. **119**, 467 (II, III).
——, 1961, Rev. Mod. Phys. **33**, 467 (III).
——, and S. Frautschi, 1961, Phys. Rev. **123**, 1478 (VIII).
——, 1964, Physics **1**, 77 (II).
——, M. Goldberger, and F. E. Low, 1968, Phys. Rev. Letters **22**, 208 (VII, X).
——, and A. Pignotti, 1968, Phys. Rev. Letters **20**, 1078 (II, VII).
——, 1969, Phys. Rev. Letters **22**, 364 (II).
——, 1970, Phys. Today **23**, 23 (II).
Childers, R. W., 1969, Phys. Rev. Letters **23**, 357 (II).
Chiu, C., and J. Finkelstein, 1968, Phys. Letters **27B**, 510 (VIII).
——, and A. Stirling, 1968, Phys. Letters **26B**, 236 (II).
Chu, S.-Y., G. Epstein, P. Kaus, R. C. Slansky, and F. Zachariasen, 1968, Phys. Rev. **175**, 2098 (II, III).
Cline, D., J. English, D. Reader, R. Terrell, and J. Twitty, 1968, Phys. Rev. Letters **21**, 1268 (X).
——, *et al.*, 1967, Phys. Rev. Letters **19**, 675 (X).
——, *et al.*, 1969, Phys. Rev. Letters **23**, 1318 (X).
Coffin, C. F., *et al.*, 1967, Phys. Rev. **159**, 1169 (X).
Cohen-Tannoudji, G., A. Morel, and H. Navelet, 1968, Ann. Phys. (N.Y.) **46**, 239 (II).
Collins, P., and R. Johnson, 1969, Phys. Rev. **182**, 1755 (X).
——, G. Ross, and E. J. Squires, 1969, Nucl. Phys. **B10**, 475 (II).
Collop, D., 1970, Nucl. Phys. **B15**, 229 (IV, VII).
Coon, D. D., 1969, Phys. Rev. **186**, 1422 (II).
Cooper, F., 1970, Phys. Rev. D **1**, 1140 (V).
Cooper, W., *et al.*, 1968, Phys. Rev. Letters **20**, 1059 (X).
Cordes, J., and P. J. O'Donnell, 1969, Phys. Rev. **185**, 1858 (X).
Costa, G., 1969, Nuovo Cimento Letters **1**, 665 (V).
Coulter, P. W., E. S. Ma, and G. L. Shaw, 1969, Phys. Rev. Letters **23**, 106 (II).
Crennell, D., *et al.*, 1968, Phys. Letters **28B**, 136 (X).
Cronin, J., and K. Kang, 1969, Phys. Rev. Letters **23**, 1004 (V).
Cronström, C., and M. Noga, 1970, Nucl. Phys. **B17**, 477 (II).
Cutkosky, R., 1963, Phys. Rev. **131**, 1888 (VIII).
Danburg, J., *et al.*, 1969, Phys. Letters **30B**, 270 (X).
Dashen, R., and S. Frautschi, 1966a, Phys. Rev. **143**, 1171; 1966b, **145**, 1287 (V).
——, and M. Gell-Mann, 1966, Phys. Rev. Letters **17**, 340 (V).
——, 1969, Phys. Rev. **183**, 1245 (V).
——, and M. Weinstein, 1969a, Phys. Rev. **183**, 1261 (V).
——, and M. Weinstein, 1969b, Phys. Rev. **188**, 2330 (V).
——, and S. Y. Lee, 1969, Phys. Rev. Letters **22**, 366 (V).
Dass, G., C. Michael, and R. Phillips, 1969, Nucl. Phys. **B9**, 544 (X).
Delbourgo, R., and P. Rotelli, 1969, Phys. Letters **30B**, 192 (VII).
Dixon, A. C., 1905, Proc. London Math. Soc. **2**, 8 (VII).
Doesch, H., and D. Gordon, 1968, Nuovo Cimento **57A**, 82 (V).
Dolen, R., D. Horn, and C. Schmid, 1968, Phys. Rev. **166**, 1768 (II, III, VII, X).
Donnachie, A., R. Kirsopp, 1969, Nucl. Phys. **B10**, 433 (II).
Dothan, Y., M. Gell-Mann, and Y. Ne'eman, 1965, Phys. Letters **17**, 148 (VII).
Drago, F., 1969, Nuovo Cimento Letters **2**, 712 (V).
——, and S. Matsuda, 1969, Phys. Rev. **181**, 2095 (III).
Drell, S. D., A. C. Finn, and M. H. Goldhaber, 1967, Phys. Rev. **157**, 1402 (V).
Durand, L., 1968, Phys. Rev. **166**, 1680 (III).
Fayyazuddin and Riazuddin, 1969, Phys. Letters **28B**, 561.
Fenster, S., and K. C. Wali, 1970, Phys. Rev. D **1**, 1409 (V, X).
Ferrari, E., and F. Selleri, 1962, Nuovo Cimento Suppl. **24**, 453 (X).
Fivel, D. I., and P. K. Mitter, 1969, Phys. Rev. **183**, 1240 (III).
Foley, K., *et al.*, 1963, Phys. Rev. Letters **11**, 503; 1965, **15**, 45 (X).
Foster, M., Ph. Gavillet, G. Labrosse, L. Montanet, R. A. Salmeron, P. Villamoes, C. Ghesquiere, and E. Lillestol, 1968, Nucl. Phys. **B6**, 107 (X).
Fox, G. C., 1967, Ph.D. thesis, Cambridge University (unpublished) (II).
——, 1969, Proceedings of the Stonybrook Conference on High Energy Physics (to be published) (VIII, X).
Frampton, P., 1969a, Phys. Rev. **D1**, 3141 (V).
——, 1969b, Phys. Letters **32B**, 195
——, and Y. Nambu, 1969, in *Essays in Theoretical Physics Dedicated to Gregor Wentzel* (University of Chicago, Chicago, 1970). (III).
Frautschi, S., M. Gell-Mann, and F. Zachariasen, 1962, Phys. Rev. **126**, 2204 (VIII).
Freedman, D., and L-L. Wang, 1967, Phys. Rev. **153**, 1596 (IV).
Freund, P. G. O., 1968a, Phys. Rev. Letters **20**, 235 (VIII).
——, 1968b, Phys. Rev. Letters **21**, 235 (VI).
——, 1969a, Phys. Rev. Letters **23**, 449 (V).
——, 1969b, Nuovo Cimento Letters **1**, 928 (VII).
——, and R. Rivers, 1969, Phys. Letters **29B**, 510 (V).
——, and E. Schonberg, 1969, Phys. Letters **28B**, 600 (V).
——, R. Waltz, and J. Rosner, 1969, Nucl. Phys. **B13**, 237 (VIII).
Frye, G., 1970a, Physics Letters **31B**, 26 (VII).
——, 1970b, Phys. Rev. D **1**, 1194 (VII).
Fubini, S., 1966, Nuovo Cimento **43A**, 475.
——, D. Gordon, and G. Veneziano, 1969, Phys. Letters **29B**, 679 (VII, IX, XI).
——, and G. Veneziano, 1969, Nuovo Cimento **64A**, 811 (V, VII, IX, X).
Fujisaki, H., 1969a, Nuovo Cimento Letters **1**, 625; 1969b, *ibid.* **2**, 41 (V).
Gallardo, J., and L. Susskind, 1970, Phys. Rev. D **1**, 1186 (VII).
——, E. Galli, and L. Susskind, 1970, Phys. Rev. D **1**, 1189 (VII).
Geffen, D., 1969, Phys. Rev. Letters **23**, 897 (V).
Gell-Mann, M., and M. Levy, 1960, Nuovo Cimento **16**, 705 (V).
——, 1961, Caltech CTSL-20, in M. Gell-Mann, M. Neeman, and Y. Neeman, *The Eightfold Way* (Benjamin, New York) (VII).
——, 1964a, Physics **1**, 63 (V).
——, 1964b, Phys. Letters **8**, 214 (VIII).
Gilman, F., and H. Harari, 1968, Phys. Rev. **165**, 1803 (II, III).
——, H. Harari, and Y. Zarmi, 1968, Phys. Rev. Letters **21**, 323 (II).
Goebel, C-J., 1968, Phys. Rev. Letters **21**, 383 (II, III).
——, M. Blackman, and K. C. Wali, 1969, Phys. Rev. **182**, 1487 (II, VIII).

——, and B. Sakita, 1969, Phys. Rev. Letters **22,** 257 (VII).
Goldberg, H., and Y. Srivastava, 1969, Phys. Rev. Letters **22,** 749 (V).
Goldberger, M. L., and S. B. Treiman, 1958, Phys. Rev. **110,** 1178 (V).
Gottfried, K., and J. D. Jackson, 1964, Nuovo Cimento **33,** 309 (X).
Green, M. B., 1970, Phys. Rev. D **1,** 450. (IX).
Gribov, V. N., and I. Ya. Pomeranchuk, 1962, Phys. Rev. Letters **9,** 238 (IV).
——, 1963, Sov. Phys. JETP **16,** 1080 (X).
Gross, D., and H. Pagels, 1968, Phys. Rev. Letters **20,** 961 (V).
——, 1969, Nucl. Phys. **B13,** 467 (VII).
Gunion, J. F., and H. J. Yesian, 1969, Phys. Rev. **186,** 1415 (X).
Gutay, L., F. T. Meire, and J. H. Scharenguivel, 1969, Phys. Rev. Letters **23,** 431 (X).
Hagedorn, R., 1968, Nuovo Cimento **56A,** 1027 (VII, VIII, XI).
Hara, Y., 1969, Phys. Rev. **182,** 1906 (IV).
Harari, H., 1968, Phys. Rev. Letters **20,** 1395 (II, VI, VII).
——, 1969, Phys. Rev. Letters **22,** 569 (II, VIII, X).
Hopkinson, J. F. L., and H-M. Chan, 1969, Nucl. Phys. **B13,** 294 (VII).
——, and E. Plahte, 1969, Phys. Letters **28B,** 489 (VII).
——, and R. G. Roberts, 1969, Nuovo Cimento Letters **2,** 466 (X).
Hsu, J-P., *et al.*, 1969, Phys. Rev. **181,** 2011 (V).
Huang, K., 1969, Phys. Rev. Letters **23,** 903 (VI, VIII).
Igi, K., 1962, Phys. Rev. Letters **9,** 76 (II, III).
——, and S. Matsuda, 1967, Phys. Rev. Letters **18,** 625 (II, III, X).
——, 1968, Phys. Letters **28B,** 330 (X).
——, 1969, Nuovo Cimento **63A,** 987 (X).
——, and J. K. Storrow, 1969, Nuovo Cimento **62A,** 829 (V, X).
Inami, T., 1969, Nuovo Cimento **63A,** 987 (X).
Jackson, J. D., 1964, Nuovo Cimento **34,** 1644 (X).
——, and H. Pilkuhn, 1964, Nuovo Cimento **33,** 906 (X).
——, and G. Hite, 1967, Phys. Rev. **169,** 1248 (II).
——, and C. Quigg, 1968, (private communication) (X).
Jacob, M., and J. Mandelbrojt, 1969, Nuovo Cimento **63A,** 279 (II, III).
Jacobs, M. A., 1969, Phys. Rev. **184,** 1574 (II).
Jengo, R., 1968, CERN Report Th. 948 (unpublished) (II).
——, 1969, Phys. Letters **28B,** 606 (II, VIII).
——, and E. Remiddi, 1969a, Nuovo Cimento Letters **1,** 637 (X).
——, and E. Remiddi, 1969b, Nuovo Cimento Letters **1,** 922 (V).
Jones, C. E., and V. Teplitz, 1967, Phys. Rev. Letters **19,** 135 (II, III).
Jones, L., and H. Wyld, 1969, Phys. Rev. Letters **23,** 814 (X).
Joshi, G., and A. Pagnamenta, 1969, Phys. Rev. **182,** 1574 (II).
Kato, S., D. Koehler, T. Novey, A. Yokasawa, and G. Burleson, 1970, Phys. Rev. Letters **24,** 615 (VIII).
Kaus, P., and F. Zachariasen, 1968, Phys. Rev. **171,** 1597 (VIII).
Kawarabayashi, K., S. Kitakado, and H. Yabuki, 1968, Phys. Letters **28B,** 432 (III, V).
——, S. Kitakado, and H. Yabuki, 1969, Nuovo Cimento Letters **1,** 564 (II).
Kernan, A., and H. Shepard, 1969, Phys. Rev. Letters **23,** 1314 (XI).
Khuri, N. N., 1963, Phys. Rev. **132,** 914 (II, III).
——, 1966, Phys. Rev. **153,** 1477 (V).
——, 1967, Phys. Rev. Letters **18,** 1094 (II).
——, 1968, Phys. Rev. **176,** 2026 (II, III).
——, 1969, Phys. Rev. **185,** 1876 (II, III).
Kikkawa, K., 1969, Phys. Rev. **187,** 2249 (IX).
——, S. Klein, B. Sakita, and M. Virasoro, 1969, Phys. Rev. **D1,** 3258 (IX).
——, B. Sakita, and M. Virasoro, 1969, Phys. Rev. **184,** 1701 (VII, IX).
Koba, Z., and H. B. Nielsen, 1969, Nucl. Phys. **B10,** 633 (VII).
Krzywicki, A., 1969, Phys. Rev. **187,** 1964 (VII).
Kugler, M., 1968, Phys. Rev. Letters **21,** 570 (III).
——, and M. Milgrom, 1969, Nucl. Phys. **B13,** 294 (VII).
Landshoff, P. V., and W. Zakrzewski, 1969, Nucl. Phys. **B12,** 216 (VII).
Lee, T. D., 1954, Phys. Rev. **95,** 1329 (I).
——, 1969, in *Essays in Theoretical Physics Dedicated to Gregor Wentzel,* (University of Chicago, Chicago, 1970) (VII, VIII, IX).
——, and G. C. Wick, 1969, Nucl. Phys. **B9,** 209 (VII, IX).
——, S. Weinberg, and B. Zumino, 1967, Phys. Rev. Letters **18,** 1029 (V).
Levy, M., and J. Sucher, 1969, Phys. Rev. **186,** 1656 (VII).
Lichtenberg, D., R. G. Newton, and E. Predazzi, 1969, Phys. Rev. Letters **22,** 1215 (II).
Lipkin, H., 1966, Phys. Rev. Letters **16,** 1015 (VIII).
Liu, Y., and S. Okubo, 1967, Phys. Rev. Letters **19,** 190 (II).
Logunov, A., L. Soloviev, and A. Tavkhelidze, 1967, Phys. Letters **24B,** 187 (II).
Lovelace, C., 1968, Phys. Letters **28B,** 265 (III, V, X).
——, 1969a, "Proceedings ANL Conference on $\pi\pi$ and $K\pi$ Interactions," p. 562 (VI, X).
——, 1969b, Nucl. Phys. **B12,** 253 (V, X).
Low, F. E., 1955, Phys. Rev. **27,** 1392 (V).
Lys, J., *et al.*, 1968, Phys. Rev. Letters **21,** 1116 (X).
Ma, Z. A., *et al.*, 1968, Report to Vienna Conference (X).
MacDowell, S., 1959, Phys. Rev. **116,** 774 (V, X).
Mandelstam, S., 1962, Ann. Phys. (N.Y.) **19,** 254 (III).
——, 1963a, b, c, Nuovo Cimento **30,** 1113, 1127, 1148 (III, V, XI).
——, 1966, *Tokyo Summer Lectures In Physics* (Benjamin, New York), pp. 14–16.
——, 1968a, Phys. Rev. **166,** 1539 (II, IV).
——, 1968b, Phys. Rev. Letters **21,** 1724 (III, IV).
——, 1969a, Phys. Rev. **183,** 1374 (IV).
——, 1969b, Phys. Rev. **184,** 1625 (VII, VIII).
——, 1970a, Phys. Rev. D **1,** 1734; 1970b, D **1,** 1745. VII.
Mandula, J., 1968, Phys. Rev. **174,** 1948 (II).
——, C. Rebbi, R. Slansky, J. Weyers, and G. Zweig, 1969, Phys. Rev. Letters **22,** 1147 (VIII).
——, and R. Slansky, 1968, Phys. Rev. Letters **20,** 1402 (II, III).
——, J. Weyers, and G. Zweig, 1969a, b, Phys. Rev. Letters **23,** 266, 627 (VIII).
——, J. Weyers, and G. Zweig, 1970, Ann. Rev. Nucl. Sci. (to be published) (VIII).
Martin, A., 1969, Phys. Letters **29B,** 431 (VI).
Mathur, V., P. Olesen, and M. A. Rashid, 1969, Nuovo Cimento **64A,** 285.
Matsuda, S., 1969a, Phys. Rev. **185,** 1811 (II).
——, 1969b, UCRL-19266 (unpublished) (II, III).
McClellan, G., N. Mistry, P. Mostek, H. Ogren, A. Osborne, A. Silverman, J. Swart, R. Talman, and G. Diambrini-Palazzi, 1969, Phys. Rev. Letters **23,** 718 (X).
McKay, D., and W. Walter, 1969, Phys. Rev. Letters **23,** 619; 1008 (E) (V).
Michael, C., and G. Dass, 1968, Phys. Rev. **175,** 1774 (X).
Misheloff, M., 1969, Phys. Rev. **184,** 1732 (VII).
Moen, I. O., and J. W. Moffat, 1970, Nuovo Cimento Letters **3,** 473 (X).
Moffat, J. W., 1969, Nuovo Cimento Letters **2,** 773 (III).
Morgan, D., and G. Shaw, 1970, Phys. Rev. D **1,** 520 (X).
Nambu, Y., 1969, Invited paper, Conference on Symmetries and Quark Models, Wayne State University (to be published) (VII).
Nelson, T. J., 1969, Phys. Rev. **184,** 1954 (III).
Neville, D., 1969, Phys. Rev. Letters **22,** 494 (VIII).
Odorico, R., 1971a, Phys. Letters **34B,** 65 (X).
——, 1971b, CERN Preprint TH1248 (to be published).
Oehme, R., 1964, in *Strong Interactions and High Energy Physics* (Oliver and Boyd, London) (III, VIII, X).
——, 1969a, Nuovo Cimento Letters **1,** 420 (II, III).
——, 1969b, Nuovo Cimento Letters **2,** 53 (II, III).
Oh, B. Y., and W. D. Walker, 1969, Phys. Letters **28B,** 564 (X).
Ohba, I., 1969, Progr. Theoret. Phys. (Kyoto) **42,** 432 (V).
Olesen, P., 1970a, Nucl. Phys. **B18,** 459 (VII, VIII).
——, 1970b, Nucl. Phys. **B18,** 473 (IX).
Olsson, M. G., 1967, Phys. Letters **26B,** 310 (II).
Orear, J., 1964, Phys. Rev. Letters **13,** 90 (XI).
——, *et al.*, 1968, Phys. Letters **28B,** 61 (X).
Osborn, H., 1969, Nuovo Cimento Letters **1,** 513 (V).
Oyanagi, Y., 1969, Progr. Theoret. Phys. (Kyoto) **42,** 898 (V).
Paciello, M., L. Sertorio, and B. Taglienti, 1969a, Nuovo Cimento **62A,** 713 (IV).
——, L. Sertorio, and B. Taglienti, 1969b, Nuovo Cimento **63A,** 1026 (VI).
Petersson, B., and N. Tornqvist, 1969, Nucl. Phys. **B13,** 629 (X).
Phillips, R. J. N., and G. Ringland, 1969, Nucl. Phys. **B13,** 274 (II).

Pinsky, S., 1969, Phys. Rev. Letters **22**, 677 (X).
Polkinghorne, J. C., 1969, Phys. Rev. **186**, 1670 (IX).
Pomeranchuk, I. Ya., 1956, Sov. Phys. JETP **3**, 306 (VIII).
——, and L. Okun', 1956, Sov. Phys. JETP **3**, 307 (VIII).
——, 1958, Sov. Phys. JETP **7**, 499 (VIII).
Pretzl, K., and K. Igi, 1969, Nuovo Cimento **63A**, 609 (X).
Rarita, W., R. Ridell, C. Chiu, and R. Phillips, 1968, Phys. Rev. **165**, 1615 (X).
Rebbi, C., and R. Slansky, 1970, Rev. Mod. Phys. **42**, 68.
Roberts, R. G., 1969, Nuovo Cimento Letters **1**, 364 (X).
——, and F. Wagner, 1969a, Phys. Letters **29B**, 423; 1969b, Nuovo Cimento **64A**, 206 (X).
Roskies, R. Z., 1968, Phys. Rev. Letters **21**, 1851 (VI).
Rosner, J., 1968, Phys. Rev. Letters **21**, 950; 1468 (E) (VII, VIII).
——, 1969a, Phys. Rev. Letters **22**, 689 (VIII).
——, 1969b, Invited paper, Eastern Theoretical Conference (to be published) (VIII).
——, C. Rebbi, and R. Slansky, 1969, Phys. Rev. **188**, 2367 (VIII).
Rothe, H. J., 1966, UCRL-17068, Ph.D. thesis (unpublished) (X).
Roy, D., and M. Suzuki, 1969, Phys. Letters **28B**, 558 (VIII).
Rubinstein, H., G. Veneziano, and M. Virasoro, 1968, Phys. Rev. **167**, 1441 (V).
Savoy, C., 1969, Nuovo Cimento Letters **1**, 263 (V).
Scheck, F., 1969, Nuovo Cimento **63A**, 1074 (IV).
Schmid, C., 1968, Phys. Rev. Letters **20**, 628 (II, III).
——, and J. Yellin, 1969, Phys. Rev. **182**, 1449 (II, VIII).
——, 1969a, Nuovo Cimento **61A**, 289 (II).
——, 1969b, Nuovo Cimento Letters **1**, 165 (VIII).
Schnitzer, H. J., 1969, Phys. Rev. Letters **22**, 1154 (V).
Schwimmer, A., 1969, Phys. Rev. **184**, 1508 (VIII).
Sciuto, S., 1969, Nuovo Cimento Letters **2**, 411 (VII).
Shapiro, J., 1969, Phys. Rev. **179**, 1345 (III).
——, and J. Yellin, 1970, Yad. Fiz. **11**, 443 (III, XI).
Singh, V., 1967, Phys. Rev. Letters **18**, 36 (V).
Sivers, D., and J. Yellin, 1969a, UCRL-18665 (unpublished) (III).
——, and J. Yellin, 1969b, Ann. Phys. (N.Y.) **55**, 107 (II, III).
Stapp, H., 1970, Nuovo Cimento **66A**, 497 (VII).
Sugawara, H., 1969, Tokyo University of Education preprint (to be published) (V).
Susskind, L., 1969, Phys. Rev. Letters **23**, 545 (VII).
——, 1970, Phys. Rev. D **1**, 1182 (VII).
Suura, H., 1969, Phys. Rev. Letters **23**, 551; 1007 (E) (V).
Suzuki, M., 1969, Phys. Rev. Letters **23**, 205 (VI).
Swart, J. J. de, 1963, Rev. Mod. Phys. **35**, 916 (VIII).
Swift, A., and R. Tucker, 1969, Phys. Rev. **186**, 1553 (II).
Takagi, F., 1969, Progr. Theoret. Phys. (Kyoto) **41**, 1555.
Tan, C-I., and J-M. Wang, 1969, Phys. Rev. **185**, 1899 (VII).
Taylor, R. E., 1967, Proceedings of the SLAC Conference on Electron and Photon Interactions, publ. SLAC (Stanford, 1967) p. 78 (V).
Thorn, C., 1970, Phys. Rev. D **1**, 1693 (IX).
——, and M. Kaku, 1970, Phys. Rev. D **1**, 2860 (IX).
Tiktopoulos, G., 1970, Phys. Letters **31B**, 138 (XI).
Tokuda, N., 1969, Progr. Theoret. Phys. (Kyoto) **42**, 641 (III).
Toller, M., 1965, Nuovo Cimento **37**, 631 (VII).
Tompkins, D., *et al.*, 1969, Phys. Rev. Letters **23**, 725 (V).
Tricomi, F., and A. Erdelyi, 1951, Pacific J. Math. **1**, 133 (III).
Trilling, G. H., 1970, Phys. Rev. Letters **24**, 179 (VIII).
Tryon, E., 1969a, Phys. Rev. D (to be published) (X).
——, 1969b, Phys. Rev. D (to be published) (X).
Van Hove, L., 1967, Phys. Letters **24B**, 183 (III).
Vecchia, P. di and F. Drago, 1969, Nuovo Cimento Letters **1**, 917 (V).
——, F. Drago, and S. Ferrara, 1969, Phys. Letters **29B**, 114 (IV).
Veneziano, G., 1968, Nuovo Cimento **57A**, 190 (II, III, IV, VII).
Virasoro, M., 1969a, Phys. Rev. Letters **22**, 37 (VII).
——, 1969b, Phys. Rev. **177**, 2309 (IV, VII).
——, 1969c, Phys. Rev. **184**, 1621 (II, X).
Wagner, F., 1969a, Nuovo Cimento **63A**, 393.
——, 1969b, Nuovo Cimento **64A**, 189.
Wang, L.-L., 1966, Phys. Rev. **142**, 1187 (II).
Weber, T. A., and S. A. Williams, 1965, J. Math. Phys. **6**, 1980 (III).
Weinberg, S., 1966, Phys. Rev. Letters **17**, 616 (V).
——, 1968, Phys. Rev. **177**, 2604 (II, V).
——, 1969, Phys. Rev. Letters **22**, 1023 (II, V).
Weisberger, W., 1966, Phys. Rev. **143**, 1302 (V).
Williams, D., 1963, UCRL-11113 (unpublished) (II).
Wong, D., 1969a, Phys. Rev. **181**, 1900 (VIII, IX).
——, 1969b, Phys. Rev. **183**, 1412 (II).
Yellin, J., 1968, UCRL-18637 (unpublished) (III).
——, 1969a, UCRL-18664 (unpublished) (III, V).
——, 1969b, Phys. Rev. **182**, 1482 (II, V).
——, 1969c, Proceedings ANL Conference on $\pi\pi$ and $K\pi$ Interactions (VIII).
——, 1969d, Institute for Advanced Study Preprint (unpublished) (XI).
Zakrzewski, W. J., 1969, Nucl. Phys. **B14**, 458 (VII).
Zee, A., 1969, Phys. Rev. **184**, 1922 (V).
Zweig, G., 1964, CERN Report Th. 402 (unpublished) (VIII).